stichting
mathematisch
centrum

AFDELING TOEGEPASTE WISKUNDE TN 91/78 SEPTEMBER
(DEPARTMENT OF APPLIED MATHEMATICS)

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INDUCED REPRESENTATIONS AND SEMIDIRECT PRODUCTS

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Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

AMS(MOS) subject classification scheme (1970): 22D10, 22D30, sec. 20C15, 43A05, 81A54.

Induced representations and semidirect products

by

H.A. van der Meer

ABSTRACT

The basic theory of induced representations is treated, for finite groups and for locally compact second countable groups. In connection with this subject the concept of imprimitivity is discussed, with an application to the notion of localizability in quantum mechanics. Finally, we demonstrate how to use induction and imprimitivity to find complete families of irreducible unitary representations of regular semidirect products by an abelian subgroup. An important example, the continuous Poincaré group, is also presented.

PREFACE

The concept of induction is a most powerful tool in representation theory. For finite groups, this method of obtaining representations of a group by means of representations of its subgroups was designed by Frobenius in 1898. In the period 1939-1950, Wigner, Bargmann and others used induction in an implicit manner, in papers which dealt with the representations of special noncompact groups, such as the Lorentz group (cf. [1],[5],[23]). It was G.W. Mackey in the years around 1950, who constructed a unified theory of induced representations for general locally compact groups, (see [12]). He also developed an extension of the important concept of imprimitivity to locally compact groups. Imprimitivity is closely related to representation theory, in particular to the theory of induced representations. Finally, Mackey showed how to apply induction and imprimitivity to obtain irreducible unitary representations of locally compact semidirect products from certain proper subgroups ("little groups"). For an important class of semidirect products(*) these results are fairly complete (see §4.3). This method, known as the little group method, had been used earlier by Wigner in connection with the Poincaré group (see [23]).

The main aim of these notes is to discuss this representation theory for locally compact semidirect products, framed by Mackey. This is done in the last chapter; the other chapters contain preliminaries. Some of the subjects in these chapters are perhaps developed somewhat out of proportion, and not always entirely associated with the last chapter. The cause of this is that we have not tried to be efficient, that is, to reach our ultimate goal as quickly as possible. On the contrary, we have been guided in some cases (e.g. quasi-invariant measures on coset spaces) by our appreciation of nice results.

The chapters I, III and IV have been used (in not quite the same form) for the colloquium "Representations of locally noncompact groups with applications", organized by the Mathematisch Centrum in the academic year 1977/1978, and will be published in the series "MC Syllabus".

(*) For instance, the Poincaré group and the Euclidean groups belong to this class.

ACKNOWLEDGEMENT

The author has the pleasant duty of thanking Dr. Tom Koornwinder for his valuable support during the preparation of these notes.

CONTENTS

I INDUCED REPRESENTATIONS OF FINITE GROUPS

1.1 Introduction

In this chapter we plan to discuss rather extensively the basic features of induction on finite groups. In the first place we aim to provide a motivation for the theory of Mackey, which is to be discussed in the next chapters. Secondly, this chapter could serve as a simple introduction for people who take interest in advanced representation theory of finite groups. For further reading in this direction we refer to the excellent exposé of SERRE [19], where among other things the important theorems of Artin and Brauer are discussed, which ensue from the induction process.

We will start with reviewing briefly some basic facts from the general representation theory of finite groups. Next the inducing construction will be presented, first for characters only (§1.3), and then for representations (§1.4). Finally we will prove a useful theorem, which provides us with a way of deciding whether an arbitrary representation is induced from a subgroup. The extension of this theorem to locally compact groups will be given in §3.1.

We emphasize that G will always denote a finite group, unless otherwise stated. Furthermore, all vector spaces are assumed to be complex and finite-dimensional.

The reader may keep in mind that many of the results apply to compact groups as well. This can be seen by replacing expressions of the form

$$\frac{1}{|G|}\sum_{x\in G} \cdots \qquad \text{by} \qquad \int_G \cdots \, dx,$$

where $|G|$ denotes the cardinality of a finite group G and dx the normalized Haar measure on a compact group G.

1.2 General representation theory for finite groups

Let V be a finite-dimensional complex vector space. By $G\ell(V)$ we will denote the group of invertible linear operators on V. A homomorphism τ from a finite group G into $G\ell(V)$ is called a representation of G on V.

Suppose that there exists a linear subspace V' of V, which is stable under the action of τ, i.e. $\tau(x)\ V' = V'$ for all x in G. Then, denoting the operators $\tau(x)$ restricted to V' by $\tau'(x)$, we obtain a new representation of G; $\tau'\colon G \to G\ell(V')$. We call τ' a subrepresentation of τ. If τ admits no non-trivial stable subspaces, then τ is said to be irreducible. Let now τ' be a subrepresentation of τ on V'. By π_0 we denote the "average" of a mapping π from V into itself with $\pi(V) = V'$ and $\pi^2 = \pi$, that is,

$$\pi_0 := \frac{1}{|G|} \sum_{x \in G} \tau(x)\pi\tau(x)^{-1}.$$

Clearly $\pi_0(V) = V'$, and one verifies easily that the complement V'' in V of V' corresponding to π_0 (i.e. $V'' = \mathrm{kernel}(\pi_0)$) is stable under τ. The sub-representation τ'' corresponding to V'' is called complementary to τ', and τ is called the direct sum of τ' and τ''. This is denoted by $\tau = \tau' \oplus \tau''$. Conversely, if we are given two representations τ and σ of G on spaces V and W respectively, we can form in an obvious way a new representation $\tau \oplus \sigma$ on the direct sum $V \oplus W$.

By iterating the construction of complementary subrepresentations given above, we see that any representation τ of G can be written as a direct sum of irreducible subrepresentations. This result is known as the theorem of Maschke. Unfortunately, such a decomposition is not always unique, as a simple counterexample may show. We will say more about this below.

Let $Rep(G)$ denote the set of all representations of G. We define an equivalence relation in $Rep(G)$ by calling $\tau,\sigma \in Rep(G)$ equivalent (notation: $\tau \simeq \sigma$) if there exists an invertible linear mapping $T\colon V(\tau) \to V(\sigma)$ such that

$$\text{(1.1)} \qquad T\tau(x) = \sigma(x)T, \qquad \forall\ x \in G.$$

(By $V(\tau)$ we denote the representation space of a representation τ.) It is clear that an equivalence class containing an irreducible representation, can contain only irreducible representations. The set of equivalence classes of irreducible representation is called the dual of G and denoted by $\hat{G}$.

Let $\tau \in Rep(G)$. The complex-valued function χ on G defined by

$$\chi(x) = \mathrm{trace}(\tau(x)), \qquad x \in G,$$

is called the character of τ. One verifies easily the following properties of characters.

LEMMA 1.1. *Let* $\tau,\sigma \in \mathcal{R}ep(g)$ *and let* χ *and* ϕ *denote their respective characters. Then*

(i) $\chi(e) =$ *dimension* $(V(\tau))$;

(ii) $\chi(x^{-1}) = \overline{\chi(x)}, \quad \forall\, x \in G$;

(iii) $\chi(yxy^{-1}) = \chi(x)$, $\forall\, x,y \in G$,

(iv) *the character of* $\tau \oplus \sigma$ *equals* $\chi + \phi$,

(v) $\tau \simeq \sigma \Rightarrow \chi = \phi$.

We continue with discussing several important consequences of this simple lemma, especially of (iii).

Two elements x and y of G are said to be conjugate if $x = zyz^{-1}$ for some z in G. This defines an equivalence relation in G, so we can partition G into equivalence classes, which are called conjugacy classes. We shall see below that the number of conjugacy classes, the so-called class number of G, is an important feature of the group G. From lemma 1.1 (iii) it follows that characters are constant on conjugacy classes. In general, we call a complex-valued function on G which satisfies this condition a class function (or central function). The set of all class functions on G, denoted by $c\ell(G)$, is a linear subspace of the space $\ell^2(G)$ of all complex-valued functions on G. The latter space can be equipped with an inner product, defined by

$$(\phi,\psi) := \frac{1}{|G|} \sum_{x\in G} \phi(x)\overline{\psi(x)}, \qquad \phi,\psi \in \ell^2(G).$$

With an irreducible character we mean the character of an irreducible representation. The set of all irreducible characters of G will be denoted by $\mathcal{I}rr(G)$. The following lemma exposes the distinguished role played by $\mathcal{I}rr(G)$ in the space $c\ell(G)$.

LEMMA 1.2. *The elements of* $\mathcal{I}rr(G)$ *form an orthonormal basis for* $c\ell(G)$.

COROLLARY 1.3. *A class function* ϕ *is a character if and only if for each* χ *in* $Irr(G)$ *the number* (ϕ,χ) *is a nonnegative integer.*

PROOF. Clear from the theorem of Maschke, mentioned above, lemma 1.1 (iv) and lemma 1.2. □

We continue this preliminary subsection with a discussion of the proof of lemma 1.2, and some of its corollaries. First we need the celebrated lemma of Schur. We will take the elements of $\hat{G}$ to be proper representations, for convenience. By virtue of lemma 1.1 (v) we can unambiguously speak about the character of $\tau \in \hat{G}$.

LEMMA 1.4. (Schur) *Let* $\tau,\sigma \in \hat{G}$, *and suppose we are given a nonzero linear mapping* $T\colon V(\tau) \to V(\sigma)$, *which satisfies*

$$T\tau(x) = \sigma(x)T, \qquad \forall\ x \in \sigma.$$

Then $\tau = \sigma$ *and* T *is a scalar multiple of the identity on the representation space.*

PROOF. The obvious observation that the kernel and the range of T are invariant subspaces for τ and σ, respectively, shows that T is either zero or invertible. In the second case we have $\tau = \sigma$. Moreover, if T is invertible and if λ is any eigenvalue of T, then iteration of the preceding argument yields $T - \lambda I = 0$, where I denotes the identity on $V(\tau) = V(\sigma)$. □

Next we choose a basis in $V(\tau)$ and in $V(\sigma)$ for $\tau,\sigma \in \hat{G}$. Then τ and σ can be written in matrix form: $\tau(x) = (\tau_{ij}(x))$ and $\sigma(x) = (\sigma_{ij}(x))$. The Schur lemma implies the following orthogonality relations between matrix elements of τ and σ.

COROLLARY 1.5.

(i) *For* $\tau \neq \sigma$ *one has*

$$\frac{1}{|G|}\sum_{x\in G} \tau_{ij}(x)\sigma_{kl}(x^{-1}) = 0, \qquad \forall\ i,j,k,l.$$

(ii) $$\frac{1}{|G|}\sum_{x\in G} \tau_{ij}(x)\tau_{kl}(x^{-1}) = \frac{1}{\dim(V(\tau))}\,\delta_{ik}\delta_{jk}.$$

PROOF. Let $T = (T_{ij})$ be a linear mapping from $V(\tau)$ into $V(\sigma)$. Then

$$T^0 := \frac{1}{|G|} \sum_{x \in G} \sigma(x^{-1}) T \tau(x)$$

is also a linear mapping from $V(\tau)$ into $V(\sigma)$. Moreover, one checks easily that T^0 satisfies relation (1.1). Since

$$\operatorname{trace}(T^0) = \frac{1}{|G|} \sum_{x \in G} \operatorname{trace}(\sigma(x^{-1}) T \tau(x)) = \operatorname{trace}(T),$$

the eigenvalues of T^0 are all equal to $(\dim V(\tau))^{-1}$. trace(T). Finally, choosing for T the matrix with $T_{rs} = \delta_{rj} . \delta_{sk}$, the identities stated in the corollary are readily verified. □

Let χ and ϕ be irreducible characters of G. After choosing the indices in the orthogonality relations stated above conveniently, we find $(\chi,\chi) = 1$ and $(\chi,\phi) = 0$ for $\chi \neq \phi$. In order to finish the proof of lemma 1.2, we have to check completeness of the system $\mathcal{Irr}(G)$ in $\mathcal{cl}(G)$.

Let $\alpha \in \mathcal{cl}(G)$, and let τ be an irreducible representation of G with character χ. The operator $\tau(\alpha)$ on $V(\tau)$ defined by

$$\tau(\alpha) = \sum_{x \in G} \alpha(x) \tau(x)$$

satisfies (1.1), and is therefore a scalar multiple of the identity on $V(\tau)$ (possibly zero). We have

$$\operatorname{trace}(\tau(\alpha)) = \sum_{x \in G} \alpha(x) \chi(x) = |G| . (\alpha, \bar{\chi}),$$

where $\bar{\chi}(x) := \overline{\chi(x)}$. Hence,

$$\tau(\alpha) = \frac{|G|}{\dim(V(\tau))} (\alpha, \bar{\chi}) . I.$$

Next, suppose $(\alpha,\bar{\chi}) = 0$ for all $\chi \in \mathcal{Irr}(G)$. Then $\tau(\alpha) = 0$ for all $\tau \in \hat{G}$. If we define $\sigma(\alpha)$ for an arbitrary representation of G, we have again $\sigma(\alpha) = 0$, by direct sum decomposition. In order to finish our argument, we need the following example.

EXAMPLE 1.6. Let λ be the representation of G on the space $\ell^2(G)$, defined by

$$(\lambda(x)f)(y) = f(x^{-1}y), \qquad f \in \ell^2(G).$$

A basis for $\ell^2(G)$ is formed by the functions $\{\varepsilon_x\}_{x\in G}$, defined by

$$\varepsilon_x(y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\lambda(x)\varepsilon_y = \varepsilon_{xy}$. The representation λ is called the left regular representation of G. The right regular representation ρ of G is defined on $\ell^2(G)$ by

$$(\rho(x)f)(y) = f(yx), \qquad f \in \ell^2(G).$$

For σ in the paragraph preceding this example we take λ. Then

$$0 = \lambda(\alpha)\varepsilon_e = \sum_{x\in G} \alpha(x)\lambda(x)\varepsilon_e = \sum_{x\in G} \alpha(x)\varepsilon_x.$$

Hence, $\alpha(x) = 0$ for all x in G. Thus, we proved that any function in $\mathcal{Cl}(G)$ which is orthogonal to the system $\{\overline{\chi}; \chi \in \mathit{Irr}(G)\}$ must be zero. Clearly this implies the same for the system $\mathit{Irr}(G)$, so we are through with lemma 1.2. This lemma has important consequences. First, note that it follows from the orthogonality relations for the irreducible characters that non-equivalent irreducible representations have different characters. This fact yields

LEMMA 1.7. *The number of non-equivalent irreducible representations of* G *equals the class number of* G.

PROOF. The cardinality of $\hat{G}$ is equal to that of $\mathit{Irr}(G)$, by the observation made above. The number of elements in $\mathit{Irr}(G)$ is, in its turn, equal to the dimension of $\mathcal{Cl}(G)$, which obviously is the class number of G. □

Next, let τ be any representation of G, and let

$$(1.2) \qquad \tau = \sigma_1 \oplus \dots \oplus \sigma_n$$

be a decomposition of τ into irreducible representations. Write $\chi, \chi_1, \dots, \chi_n$

for the characters of $\tau,\sigma_1,\ldots,\sigma_n$, respectively. The following lemma establishes the degree of uniqueness of decomposition (1.2).

LEMMA 1.8. *The number of* σ_j *equivalent to a certain* σ_i $(1 \le i,j \le n)$ *is equal to the number* (χ,χ_i). *In particular, it does not depend on the chosen decomposition.*

PROOF. We have $(\chi,\chi_i) = \sum_{j=1}^{n} (\chi_j,\chi_i)$, and the result follows from the orthonormality relations for irreducible character. □

The character χ of the regular representation ρ is readily found to be given by $\chi(e) = |G|$ and $\chi(x) = 0$ if $x \neq e$. Let ψ be an irreducible character of G. Then

$$(\chi,\psi) = \frac{1}{|G|} \sum_{x\in G} \chi(x)\overline{\psi(x)} = \psi(e).$$

Hence, each τ in $\hat{G}$ occurs in the direct sum decomposition of ρ, with multiplicity equal to $\dim(V(\tau))$ (we call the number of subrepresentations equivalent to a given irreducible representation τ, occuring in a representation σ, the multiplicity of τ in σ). This observation implies the following lemma.

LEMMA 1.9.

$$\sum_{\tau\in\hat{G}} (\dim(V(\tau)))^2 = |G|.$$

PROOF. $\mathrm{Dim}(\ell^2(G)) = |G|$. □

Last but not least we notice that the converse of lemma 1.1 (v) follows from lemma 1.1. Thus we have

LEMMA 1.10. *Two representations of* G *are equivalent if and only if they have the same character.*

EXAMPLE 1.11. Let $S_3 = \{(1),(12),(13),(23),(123),(132)\}$ be the permutation group of an ordered set of three elements. This group is isomorphic to the dihedral group D_3, which consists of those rotations and reflections of the real plane that preserve a regular triangle. If we set $s = (12)$ and $r = (123)$, we get $s^2 = (1) = e$, $r^3 = e$, $sr = r^2s$ and $rs = sr^2$. The conjugacy classes are readily seen to be $K_1 = \{e\}$, $K_2 = \{s,sr,rs\}$ and $K_3 = \{r,r^2\}$. Hence, there are three irreducible characters. Furthermore, we must have

$$\sum_{\tau\in\hat{S}_3} (\dim \tau)^2 = \sum_{\chi\in \mathit{Irr}(S_3)} (\chi(1))^2 = |S_3| = 6.$$

Therefore, two of the irreducible characters are one-dimensional and one is two-dimensional. Let χ_1 be the trivial character ($\chi_1 = 1$) and let χ_2 be the one-dimensional character that can be defined on all permutation groups: $\chi_2(x) = 1$ if x is even and $\chi_2(x) = -1$ if x is odd (we call a permutation even (odd) if it contains an even (odd) number of inversions). For S_3 we get $\chi_2(K_2) = -1$ and $\chi_2(K_3) = 1$, denoting by $\chi(K)$ the constant value of χ on a conjugacy class K. The third character can now be reconstructed from the orthogonality relations, knowing that $\chi_3(e) = 2$:

$$(\chi_1,\chi_3) = \frac{1}{6}\,(2 + 3\chi_3(K_2) + 2\chi_3(K_3)) = 0,$$

$$(\chi_2,\chi_3) = \frac{1}{6}\,(2 - 3\chi_3(K_2) + 2\chi_3(K_3)) = 0.$$

Hence, $\chi_3(K_2) = 0$ and $\chi_3(K_3) = -1$. It is convenient to store our knowledge in a so-called *character table*, that is, a matrix, with at the ij-th place the value of the i-th character on the j-th conjugacy class.

	K_1	K_2	K_3
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

The representation τ_3 corresponding to χ_3 can be realized in $\mathbb{C}^2$ by the aforementioned isomorphism of S_3 on D_3. Choosing the regular traingle conveniently in $\mathbb{R}^2 \subset \mathbb{C}^2$, we obtain as the generators of D_3 two matrices X and Y which are given by

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix},$$

respectively a reflection and a rotation through an angle $\frac{2}{3}\pi$. Clearly s corresponds with X and r with Y. Hence, $\tau_3(s) = X$, $\tau_3(s^2) = \tau_3(e) = I$ (the identity matrix), $\tau_3(r) = Y$, $\tau_3(r^2) = Y^2$, $\tau_3(sr) = XY$, $\tau_3(rs) = YX$.

The group S_3 contains an invariant subgroup of index two, namely $A_3 := \{e,r,r^2\}$, the so-called alternating group, which contains all even permutations. This subgroup is cyclic, and its character table is easily verified to be

	$\{e\}$	$\{r\}$	$\{r^2\}$
ψ_1	1	1	1
ψ_2	1	ω	ω^2
ψ_3	1	ω^2	ω

where $\omega = e^{\frac{2i\pi}{3}}$. Note that it is in general not true that a subgroup inherits the conjugacy class structure from the original group.

1.3. Induction of characters

Restricting representations of G to a subgroup H yields representations of H, with the same representation space. In general this restriction can not be reversed, that is, it is not always possible to extend representations of H to representations of G with the same representation space. For instance, the representations of A_3 corresponding to its nontrivial irreducible characters (example 1.11) cannot be extended to one-dimensional representations of S_3. However, there is a canonical construction which assigns a representation of G to every representation of H, and which is in some sort dual to the process of restriction. It proceeds by extending the representation space of a given representation of H to a larger space in which a representation of G can be defined (§1.4). For the sake of clarity we will show by means of characters that such a construction is possible, before discussing it in detail. The sense of duality in this context is to be explained at the end of this subsection.

Thus, let τ be a representation of H and let χ be its character. We shall show how χ can be extended to a character of G. The most natural way, perhaps, would be to produce a function $\dot{\chi}: G \to \mathbb{C}$ by the following definition:

$$\dot{\chi}(x) := \begin{cases} \chi(x) & \text{if } x \in H \\ 0 & \text{otherwise.} \end{cases}$$

Unfortunately, this yields in general not even a class function: take for example any irreducible character of $A_3 \subset S_3$. Another possible step is to *centralize* $\dot{\chi}$:

$$\hat{\chi}(x) := \frac{1}{|G|} \sum_{y \in G} \dot{\chi}(y^{-1}xy). \tag{1.3}$$

Here we have a class function on G, but is it a character? To check this we compute its Fourier coefficients in the space $C\ell(G)$. Let ϕ be in $Irr(G)$. By $(-,-)_G$ and $(-,-)_H$ we denote the inner products in $C\ell(G)$ and $C\ell(H)$, respectively.

$$\begin{aligned}
(\hat{\chi},\phi)_G &= \frac{1}{|G|} \sum_{x \in G} \hat{\chi}(x)\overline{\phi(x)} = \\
&= \frac{1}{|G|} \sum_{x \in G} \Big(\frac{1}{|G|} \sum_{y \in G} \dot{\chi}(y^{-1}xy)\Big)\overline{\phi(x)} = \\
&= \frac{1}{|G|^2} \sum_{x,y \in G} \dot{\chi}(y^{-1}xy)\overline{\phi(y^{-1}xy)} = \\
&= \frac{1}{|G|} \sum_{x \in G} \dot{\chi}(x)\overline{\phi(x)} = \\
&= \frac{1}{|G|} \sum_{x \in H} \chi(x)\overline{\phi(x)} = \\
&= \frac{|H|}{|G|} (\chi,\phi|_H)_H.
\end{aligned}$$

Here $\phi|_H$ denotes the character of H obtained by restricting ϕ. From corollary

1.3 we see that taking $(|G|/|H|)\cdot\hat{\chi}$ instead of $\hat{\chi}$ yields a character of G. Denoting this character by χ^G, it follows from (1.3) that

$$\chi^G(x) = \frac{1}{|H|} \sum_{y \in G} \dot{\chi}(y^{-1}xy), \qquad x \in G. \tag{1.4}$$

DEFINITION 1.12. The character χ^G defined by (1.4) is said to be *induced* on G by χ. The corresponding representation is denoted by τ^G. It is also called induced on G (by τ).

PROPOSITION 1.13. (Frobenius reciprocity theorem). *If χ and ϕ are characters of H and G respectively, H being a subgroup of G, then*

$$(\chi^G, \phi)_G = (\chi, \phi|_H)_H. \tag{1.5}$$

The proof of this proposition follows directly from the above computation, in which we did not use the irreducibility of ϕ. It provides us with information about the decomposition of χ^G when χ is irreducible. For, suppose that

$$\chi^G = \sum_{\psi \in \mathit{Irr}(G)} m_{\chi,\psi}\, \psi \quad \text{and} \quad \phi|_H = \sum_{\eta \in \mathit{Irr}(H)} n_{\phi,\eta}\, \eta.$$

Then one has for all ϕ in $\mathit{Irr}(G)$ and all χ in $\mathit{Irr}(H)$

$$m_{\chi,\phi} = (\chi^G, \phi)_G = (\chi, \phi|_H)_H = n_{\phi,\chi}.$$

Hence, we find the following corollary to proposition 1.13:

COROLLARY 1.14. *If τ and σ are irreducible representations of H and G, respectively, then the multiplicity of G in τ^G equals the multiplicity of τ in $\sigma|_H$.*

Using formula (1.4) the reader will find no difficulty in verifying the following results:

PROPOSITION 1.15. *Let χ and ϕ be characters of the subgroup $H \subset G$. Then*

$$(\chi + \phi)^G = \chi^G + \phi^G$$

and, if ψ *is a character of* G,

$$\chi^G \psi = (\chi \cdot \psi|_H)^G.$$

COROLLARY 1.16. *For representations* τ *and* σ *of* H *and a representation* ν *of* G, *one has*

$$(\tau \oplus \sigma)^G \simeq \tau^G \oplus \sigma^G$$

and

$$\tau^G \otimes \nu \simeq (\tau \otimes \nu|_H)^G.$$

COROLLARY 1.17. *If the induced representation* τ^G *is irreducible, then* τ *is irreducible.*

Unfortunately, the converse of this statement is in general false (cf. example 1.19).

PROPOSITION 1.18. (Induction in stages). *If* H_1 *and* H_2 *are subgroups of* G *such that* $H_1 \subset H_2$, *and if* τ *is a representation of* H_1, *then*

$$(\tau^{H_2})^G \simeq \tau^G.$$

REMARK.

(i) If n is the dimension of a representation τ of H, then the dimension of τ^G is n.d, where d is the index of H in G, that is, the number of different left H-cosets. This follows from (1.4).

(ii) We can define a linear mapping

$$\mathcal{R}es_H\colon C\ell(G) \to C\ell(H),$$

which sends a class function on G to its restriction to H. Formula (1.4) may be considered as a definition of ϕ^G for all ϕ in $C\ell(G)$, and the resulting mapping $\phi \to \phi^G$:

$$Ind^H\colon C\ell(H) \to C\ell(G)$$

is then linear, and, moreover, it is the adjoint of $\mathcal{R}es_H$ by (1.5). In this sense, restriction and induction are dual actions.

EXAMPLE 1.19. If we take $H \subset G$ to be the trivial subgroup $\{e\}$, and if we induce the trivial one-dimensional representation of $\{e\}$ (denote it by 1_e), then we obtain

$$1_e^G(x) = \begin{cases} |G| & \text{if } x = e \\ 0 & \text{otherwise.} \end{cases}$$

This is just the character of the regular representation of G. Application of proposition 1.18 shows that induction of the regular representation of any subgroup results in the regular representation of G.

EXAMPLE 1.20. Consider the subgroup A_3 of S_3 discussed in §1.2. Inducing the character ψ_2 of A_3 on S_3 yields

$$\psi_2^{S_3}(K_1) = 2, \quad \psi_2^{S_3}(K_2) = 0 \quad \text{and} \quad \psi_2^{S_3}(K_3) = -1,$$

since $1 + \omega + \omega^2 = 0$, $\omega = e^{\frac{2i\pi}{3}}$. Thus we obtain the only irreducible character of S_3 of dimension greater than one. In general we call a group *monomial* whenever all its irreducible representations are induced by one-dimensional representations.

EXAMPLE 1.21. Suppose that there are two subgroups N and H of G, such that

(i) N is invariant,
(ii) $G = N \cdot H$ and
(iii) $N \cap H = \{e\}$.

Then G is called a *semidirect product* (of N and H). Note that (ii) and (iii) imply that every element of G can be written uniquely as the product of an element of N and an element of H. If the additional condition

(iv) N is commutative

is satisfied, then G enjoys the property of having all of its irreducible representations induced from subgroups of the form $N \cdot H'$, where H' is a subgroup of H (a *little group*). This is also true for infinite locally compact semidirect products satisfying (iv), be it under a certain restriction of a measure theoretical kind. We will come to this in chapter IV. Note that $S_3 = A_3 \cdot \{e,s\}$ is an example of a semidirect product.

1.4. The inducing construction

We will now explicitly construct the representation τ^G, induced by a given representation τ of a subgroup H of G. First we define a representation $\hat{\tau}$ of G in terms of τ and then we prove that its character equals χ^G, where χ is the character of τ. Except for a lot of technical complications of a mainly measure theoretical kind, the following procedure is the same as that for locally compact groups.

Let $V = V(\tau)$ be the representation space of τ. Define F_τ as the linear space of all functions f: $G \to V$ that satisfy

$$(1.6) \qquad f(xy) = \tau(y^{-1})f(x), \qquad (\forall x \in G, \forall y \in H).$$

In F_τ we define an action $\hat{\tau}(y)$ for y in G, by

$$(1.7) \qquad (\hat{\tau}(y)f)(x) := f(y^{-1}x), \ (f \in F_\tau)$$

Obviously, for all y in G and all f in F_τ the new function $\hat{\tau}(y)f$ belongs to F_τ as well. Moreover, $\hat{\tau}(e)$ is the identity and, for all x, y and z in G:

$$(\hat{\tau}(y)\hat{\tau}(z)f)(x) = (\hat{\tau}(z)f)(y^{-1}x) = f(z^{-1}y^{-1}x) =$$

$$= f((yz)^{-1}x) = (\hat{\tau}(yz)f)(x).$$

In particular, it follows that $(\hat{\tau}(y))^{-1} = \hat{\tau}(y^{-1})$, so $\hat{\tau}(y)$ is invertible for all y in G. Hence $\hat{\tau}$ is a homomorphism of G into the group of all invertible linear mappings of F_τ into itself. Consequently, $\hat{\tau}$ is a representation of G.

Fix a set of representatives of left H-cosets xH, say $\{x_i\}_{i=1}^d$, with $d = |G/H|$; the index of H in G. Thus, $G = x_1H \cup \ldots \cup x_dH$, and $x_iH \cap x_jH = \emptyset$ if $i \neq j$. Clearly the functions in F_τ are determined by their values on the x_i. Hence, the mapping $f \to (f(x_1),\ldots,f(x_d))$ defines a vector space isomorphism from F_τ onto $V^d = V \oplus \ldots \oplus V$. In order to compute the character of $\hat{\tau}$, it is convenient to lift the action of $\hat{\tau}$ on F_τ to an action on V^d, also denoted by $\hat{\tau}$, by means of this isomorphism. The action of $\hat{\tau}(y)$ on V^d can be represented by a d×d array $(\hat{\tau}_{ij}(y))$ of operators on V. That is, for all y in G we have

$$(1.8)\qquad (\hat{\tau}(y)f)(x_i) = \sum_{j=1}^{d} \hat{\tau}_{ij}(y)f(x_j).$$

Let now x_ℓ be the representative of the coset containing $y^{-1}x_i$. Then $x_\ell^{-1}y^{-1}x_i \in H$, or, saying it in a different way, $x_j^{-1}y^{-1}x_i \in H$ if and only if $j = \ell$. Hence, using (1.6) and (1.7) we obtain

$$(\hat{\tau}(y)f)(x_i) = f(y^{-1}x_i) = f(x_\ell x_\ell^{-1}y^{-1}x_i) =$$

$$= \tau(x_i^{-1}yx_\ell)f(x_\ell) = \sum_{j=1}^{d} \dot{\tau}(x_i^{-1}yx_j)f(x_j),$$

where

$$\dot{\tau}(x) = \begin{cases} \tau(x) & \text{if } x \in H \\ 0 & \text{otherwise.} \end{cases}$$

Combining this with (1.8) yields $\hat{\tau}_{ij}(y) = \tau(x_i^{-1}yx_j)$. We are now in a position to compute the trace of $\hat{\tau}(y)$.

$$\text{trace } (\hat{\tau}(y)) = \sum_{i=1}^{d} \text{trace } (\hat{\tau}_{ii}(y)) =$$

$$= \sum_{i=1}^{d} \text{trace } (\dot{\tau}(x_i^{-1}yx_i)) =$$

$$\sum_{i=1} \dot{\chi}(x_i^{-1}yx_i).$$

Since $\chi(z^{-1}yz) = \chi(y)$ for all $y \in G$ and all $z \in H$, we may rewrite this expression as

$$\sum_{i=1}^{d} \frac{1}{|H|} \sum_{z\in H} \dot{\chi}(z^{-1}x_i^{-1}yx_iz).$$

If i runs from 1 to d and z runs through H, x_iz runs precisely once through G, so we have

$$\text{trace } (\hat{\tau}(y)) = \frac{1}{|H|} \sum_{x\in G} \dot{\chi}(x^{-1}yx) = \chi^G(y).$$

Since the trace of the lifted operator $\hat{\tau}(y)$ equals the trace of $\hat{\tau}(y)$ in

F_τ, we have proved that $\hat{\tau} \simeq \tau^G$.

REMARK. Formula (1.7) defines an action similar to the left regular representation, be it in a different space. If we take $H = \{e\}$ and for τ the trivial representation of H, we get $F_\tau = \ell^2(G)$. Hence the above construction is in fact a generalization of the regular representation. Generalizing in the same way the right regular representation we obtain an alternative approach.

$$(1.6)' \qquad f(yx) = \tau(y)f(x) \qquad (\forall y \in H, \forall x \in G)$$

and

$$(1.7)' \qquad (\hat{\tau}'(y)f)(x) = f(xy), \quad (x,y \in G).$$

However, it is easily verified that $\hat{\tau}$ and $\hat{\tau}'$ are equivalent. If we take H to be an arbitrary subgroup of G, we can also induce the trivial representation. In that case we have that $F_\tau = \ell^2(G/H)$, the space of all complex-valued functions on G which are constant on left cosets of H. The induced representation acts in this space just as the left regular representation. It is often called the permutation representation of G corresponding to H.

EXAMPLE 1.12. Let τ be the representation of $A_3 \subset S_3$ corresponding to the character ψ_2 (example 1.11). Note that $\tau(x) = \psi_2(x)\cdot 1_{\mathbb{C}}$ for all $x \in A_3$, since ψ_2 is a one-dimensional character. We will construct τ^{S_3} explicitly.

Choosing e and s as representatives of the left A_3-cosets in S_3, we can identify F_τ with $\mathbb{C}^2$, by sending $f \in F_\tau$ to $(f(e),f(s)) \in \mathbb{C}^2$. Using (1.6) and (1.7), the action of τ^{S_3} on $\mathbb{C}^2$ can be computed:

$$\begin{cases} (\tau^{S_3}(e)f)(e) = f(e) \\ (\tau^{S_3}(e)f)(s) = f(s) \end{cases},$$

$$\begin{cases} (\tau^{S_3}(s)f)(e) = f(s) \\ (\tau^{S_3}(s)f)(s) = f(e) \end{cases},$$

and

$$\begin{cases} (\tau^{S_3}(r)f)(e) = f(r^2) = \tau(r)f(e) = \omega f(e) \\ (\tau^{S_3}(r)f)(s) = f(r^2 s) = f(sr) = \tau(r^2)f(s) = \omega^2 f(s). \end{cases}$$

In the same way one finds

$$\tau^{S_3}(r^2)\colon (f(e),f(s)) \to (\omega^2 f(e),\omega f(s)),$$

$$\tau^{S_3}(sr)\colon (f(e),f(s)) \to (\omega^2 f(s),\omega f(e))$$

and

$$\tau^{S_3}(rs)\colon (f(e),f(s)) \to (\omega f(s),\omega^2 f(e)).$$

Hence, with respect to the basis (1,0), (0,1) of $\mathbb{C}^2$, we can realize τ^{S_3} as follows:

$$\tau^{S_3}(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau^{S_3}(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^{S_3}(r) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix},$$

$$\tau^{S_3}(r^2) = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}, \quad \tau^{S_3}(sr) = \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}, \quad \tau^{S_3}(rs) = \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}.$$

This unitary representation is clearly equivalent to the one we presented in example 1.11, which we had called τ_3.

1.5. Finite systems of imprimitivity

We start this section with some preliminary remarks on so-called G-spaces. Suppose that we are given a (not necessarily finite) group G, and a set Γ on which G acts in the following way. Each $x \in G$ defines a bijection $\gamma \to x(\gamma)$ of Γ such that (i) $e(\gamma) = \gamma$ for all $\gamma \in \Gamma$ and (ii) $x(y(\gamma)) = (xy)(\gamma)$ for all $x,y \in G$. Then Γ is said to be a G-*space*. Furthermore, Γ is said to be a *trivial* G-space if each mapping $\gamma \to x(\gamma)$ is the identity on Γ. It is called a *transitive* G-space if for any pair $\gamma,\gamma' \in \Gamma$ there exists an $x \in G$ with $x(\gamma) = \gamma'$. An example of this situation is provided by taking $\Gamma = G/H$, where H is a subgroup of G. For, let the G-action be defined by $y\colon xH \to y(xH) := (yx)H$. Obviously, G/H is a transitive G-space. On the other hand, any transitive G-space can be written as G/H for some subgroup H. Indeed, fix $\gamma_0 \in \Gamma$ and let H be the stabilizer in G of γ_0, that is,

$$H := \{x \in G;\ x(\gamma_0) = \gamma_0\}.$$

Then f: $xH \to x(\gamma_0)$ is a well-defined bijection from G/H onto Γ, such that for all x and y in G $f(y(xH)) = y(f(xH))$.

From now on we assume again that G is a finite group. Let $V = V(\tau)$ be the representation space of a representation τ of G. Suppose that there exist a G-space Γ, and a family of linear subspaces of V, indexed by Γ, say $\{V_\gamma\}_{\gamma\in\Gamma}$, with

(i) $$V = \sum_{\gamma\in\Gamma}^{\oplus} V_\gamma$$ (as a vector space direct sum), and

(ii) $$\tau(x)V_\gamma = V_{x(\gamma)} \qquad (\forall x \in G,\ \forall\gamma \in \Gamma)$$

(i.e. the spaces V_γ are permuted by the action of τ in V). Then we will call this family $\{V_\gamma\}_{\gamma\in\Gamma}$ a *system of imprimitivity* (s.o.i.) for τ. In that case, we say that τ admits a s.o.i. Moreover, we will call the system trivial or transitive according to Γ being a trivial or transitive G-space. It will turn out that we can obtain a lot of information about τ by means of the systems of imprimitivity admitted by τ.

For instance, it is clear that if τ admits no s.o.i. except the obvious one in which Γ has only one element, then τ is irreducible. Indeed, any direct sum decomposition of V in τ-invariant subspaces forms a (trivial) s.o.i. Such representations are often called *primitive*. It is in general not true that irreducibility implies primitivity.

<u>EXAMPLE 1.13</u>. Consider the left regular representation λ in $\ell^2(G)$. Define subspaces of $\ell^2(G)$ by

$$x \in G: \qquad \ell^2_x(G) := \{f \in \ell^2(G);\ f(y) = 0 \ \text{ if } \ y \neq x\}.$$

Clearly we have $\ell^2(G) = \Sigma^{\oplus}_{x\in G}\ \ell^2_x(G)$. Moreover,

$$\lambda(y)\ell^2_x(G) = \ell^2_{yx}(G) \qquad (\forall y,x \in G).$$

Hence we have a s.o.i. for λ with $\Gamma = G$, and the action of G on itself is defined by left multiplication with a fixed element. Obviously, this system is transitive.

The next theorem is the so-called imprimitivity theorem, stated here for finite groups.

THEOREM 1.14. *Let* τ *be a representation of* G. *The following statements are equivalent.*

(i) τ *admits a transitive system of imprimitivity.*

(ii) *There exist a subgroup* $H \subset G$ *and a representation* σ *of* H *such that* τ *is equivalent to* σ^G.

PROOF. (ii) $\Rightarrow$ (i). Suppose that $\tau = \sigma^G$. Let $\Gamma = G/H$, and denote the elements of Γ by $\bar{x} := xH$. Consider for each $\bar{x} \in \Gamma$ the subspaces $F_{\bar{x}}$ of F_σ defined by

$$F_{\bar{x}} := \{f \in F_\sigma \quad ; \; f(y) = 0 \quad \text{if} \quad y \notin \bar{x}\}.$$

As mentioned above, Γ is a transitive G-space, under the action $y\bar{x} := \overline{yx}$. Furthermore, it is clear that $y^{-1}z \notin \bar{x}$ iff $z \notin \overline{yx}$, for all x, y and z in G. Hence,

$$\tau(y)F_{\bar{x}} = \sigma^G(y)F_{\bar{x}} = F_{\overline{yx}}.$$

Finally, we have $F_\sigma = \Sigma^{\oplus}_{\bar{x}\in\Gamma} F_{\bar{x}}$.

(i) $\Rightarrow$ (ii). Let τ be a representation of G, admitting a transitive s.o.i., say $\{V_\gamma\}_{\gamma\in\Gamma}$. Then Γ can be identified with G/H, where H is a subgroup of G, stabilizing some fixed point $\gamma_0 \in \Gamma$. Accordingly, we may write $\Gamma = \{x_1 = e, x_2, \ldots, x_d\}$, if $\{x_i\}_{i=1}^d$ is a fixed set of left H-coset representatives. The identity $y(\gamma) = \gamma'$ reduces to $y(x_i) = x_j$, where $\gamma = x_i\gamma_0$ and $\gamma' = x_j\gamma_0$. Thus, $y(x_i) = x_j$ if and only if $x_j^{-1}yx_i \in H$.

Since every $\tau(x)$ is an isomorphism of $V(\tau)$ we can conclude from the transitivity of the system that all spaces V_{x_i} have the same dimension, say n. Hence, $\tau(y)$ may be written as a d×d-array of n-dimensional linear mappings

$$\tau_{ij}(y) := \tau(y)\Big|_{V_{x_i}} : V_{x_i} \to V_{x_j}.$$

Obviously, $\tau_{ij}(y)$ is the zero mapping if $y(x_i) \neq x_j$, or, equivalently, if

$x_j^{-1}yx_i \notin H$. Therefore, in order to compute the trace of $\tau(y)$, we only have to take into account $\tau_{ii}(y)$ for those values of i for which $x_i^{-1}yx_i \in H$. Furthermore, clearly $\tau_{ii}(y)$ and $\tau_{11}(x_i^{-1}yx_i)$ have the same trace.

Let a representation σ of H be defined by $\sigma(y) := (\tau|_H(y))|V_{x_1}$, $y \in H$ (so $V(\sigma) = V_{x_1}$). Using the preceding paragraph we can make the following computation.

$$
\begin{aligned}
\text{trace } (\tau(y)) &= \sum_{i=1}^{d} \text{trace } (\tau_{ii}(y)) = \\
&= \sum_{x_i^{-1}yx_i \in H} \text{trace } (\tau_{11}(x_i^{-1}yx_i)) = \\
&= \sum_{i=1}^{d} \text{trace } (\dot{\sigma}(x_i^{-1}yx_i)) = \\
&= \frac{1}{|H|} \sum_{z \in G} \text{trace } (\dot{\sigma}(z^{-1}yz)) = \\
&= \frac{1}{|H|} \sum_{z \in G} \dot{\chi}(z^{-1}yz),
\end{aligned}
$$

where χ is the character of σ. Hence $\sigma^G \simeq \tau$. □

COROLLARY 1.25. *All irreducible representations of* G *are induced by primitive representations.*

PROOF. A s.o.i. admitted by an irreducible representation is necessarily transitive. Therefore, the result follows via complete induction from the imprimitivity theorem 1.24, the stages theorem 1.18 and corollary 1.17. □

REMARK. The imprimitivity theorem gives rise to an alternative definition of induced representations, which is, however, less constructive than the one we used. In order to deepen the insight into the inducing process, we will make a few remarks on this different approach.

Let τ be a representation of G in a space $V = V(\tau)$. Suppose that we are given a subgroup $H \subset G$ and a linear subspace $W \subset V$, such that

(i) $\quad \tau(x)W = W, \qquad (\forall x \in H),$

and

(ii) $\qquad V = \sum_{i=1}^{d}{}^{\oplus}\, \tau(x_i)W, \qquad$ where $G/H = \{x_iH\}_{i=1}^{d}$.

Then we shall say that τ is induced by $\sigma := (\tau|_H)|_W$ (cf. SERRE [19, chapter 7]). The lack of constructiveness is easily repaired.

Indeed, let σ be a representation of $H \subset G$, in a space $W = W(\sigma)$. Consider the tensor product $\ell^2(G) \otimes W$, of the space of all complex-valued functions on G, and W. For f in $\ell^2(G)$ we define two new functions on $\ell^2(G)$, ${}_yf$ and f_y, by

$${}_yf(x) := f(y^{-1}x) \quad \text{and} \quad f_y(x) := f(xy).$$

In $\ell^2(G) \otimes W$ we define the equivalence relation $\sim$ as follows.

$$f \otimes v \sim g \otimes w \quad \text{if for some } y \in H: \begin{cases} g = f_y \\ w = \sigma(y)v. \end{cases}$$

The space of equivalence classes is denoted usually by $\ell^2(G) \otimes_H W(\sigma)$. Writing $f \otimes v$ for the equivalence class containing this element, a representation τ of G can be defined in this space by

$$\tau(y)(f\otimes v) := {}_yf \otimes v, \quad f \otimes V \in \ell^2(G) \otimes_H W(\sigma).$$

It is readily verified that τ is equivalent to σ^G.

II INDUCED REPRESENTATIONS OF LOCALLY COMPACT SECOND COUNTABLE*) GROUPS

2.1. Introduction

In this chapter and the subsequent ones, we will be concerned with unitary representations of locally compact groups, satisfying the second axiom of countability. First we recall some facts from general representation theory. Let G be a lcsc. group, fixed throughout this subsection.

*) We will abbreviate the cumbersome expression "locally compact second countable" by lcsc. throughout these notes.

A unitary representation τ of G on a Hilbert space H (which will always be assumed to be separable) is a continuous homomorphism from G into the unitary group $U(H)$ of H. Here continuity means strong continuity, i.e. the mapping $x \mapsto \tau(x)\,\xi$ on G must be continuous for all ξ in H. At once we state a useful lemma.

<u>LEMMA 2.1</u>. *Let τ be a homomorphism from* G *into the unitary group of a Hilbert space* H. *Then the following statements are equivalent:*

(i) τ *is strongly continuous;*

(ii) τ *is weakly continuous (i.e. the function* $x \mapsto (\tau(x)\xi,\eta)$ *on* G *is continuous for all* ξ *and* η *in* H*);*

(iii) *the function* $x \mapsto (\tau(x)\xi,\xi)$ *is continuous for all* ξ *in* H.

<u>PROOF</u>. Obviously (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). We show (iii) $\Rightarrow$ (i). For all x,y in G and ξ in H, we have

$$\begin{aligned}\|\tau(x)\xi - \tau(y)\xi\|^2 &= (\tau(x)\xi,\tau(x)\xi) + (\tau(y)\xi,\tau(y)\xi) - \\ &\qquad - 2\mathrm{Re}(\tau(x)\xi,\tau(y)\xi) \\ &\le 2|(\xi,\xi) - (\tau(x)\xi,\tau(y)\xi)| \\ &= 2|(\xi,\xi) - (\tau(y^{-1}x)\xi,\xi)|.\end{aligned}$$

Hence, by virtue of (iii), $\|\tau(x)\xi - \tau(y)\xi\| \to 0$ if $y \to x$. $\square$

<u>REMARK</u>. By virtue of lemma 8.28 in VARADARAJAN [21], lemma 2.1 can be sharpened as follows: A homomorphism $\tau\colon G \to U(H)$ is continuous if and only if the function $x \to (\tau(x)\xi,\xi)$ on G is measurable for each ξ (with respect to the natural Borel structures on G and $\mathbb{C}$).

Let τ be a unitary representation of G on $H = H(\tau)$. Then τ is said to be *irreducible* if there exists no closed linear subspace H' of H with $\tau(x)H' \subset H'$ for all x in G, except for the trivial ones $H' = \{0\}$ and H itself.

Let τ_1 and τ_2 be unitary representations of G on H_1 and H_2, respectively. A bounded linear operator $T\colon H_1 \to H_2$ which satisfies

$$T\tau_1(x) = \tau_2(x)T, \qquad \forall x \in G,$$

is called an *intertwining operator* for τ_1 and τ_2. The linear space of all such operators is denoted by $I(\tau_1,\tau_2)$, and, if $\tau_1 = \tau_2 = \tau$, by $I(\tau)$. In this case $I(\tau)$ can be shown to be a weakly closed $*$-algebra in the space of all bounded operators on $\mathcal{H}(\tau)$, and it is called the commuting algebra of τ. It can be shown that τ will be irreducible if and only if $I(\tau)$ contains only scalar multiples of the identity in $\mathcal{H}(\tau)$, i.e. $I(\tau) = \{\lambda I;\lambda \in \mathbb{C}\}$. This is a generalization of the well-known and easy to prove Schur lemma for finite-dimensional representations. The proof in the case of infinite dimensional unitary representations is based on the spectral theorem.

Two unitary representations τ_1 and τ_2 of G are called equivalent if $I(\tau_1,\tau_2)$ contains an isometrical isomorphism.

2.2. Homogeneous spaces

Let Γ be a locally compact second countable (lcsc.) topological space satisfying the Hausdorff separation axiom, and let G be a lcsc. group. Then Γ is called a continuous G-space if (i) Γ is a G-space (as defined in §1.5) and (ii) the mapping $(x,\gamma) \to x(\gamma)$ from $G \times \Gamma$ onto Γ is continuous. Note that this implies that each mapping $\gamma \to x(\gamma)$ is a homeomorphism from Γ onto itself. We shall say that G acts continuously on Γ. If the G-action is both continuous and transitive then Γ is called a *homogeneous space* of G. Two continuous G-spaces Γ and Δ are said to be G-homeomorphic if there exists a homeomorphism ϕ from Γ onto Δ which respects the G-action, that is, $\phi(x(\gamma)) = x(\phi(\gamma))$ for all x in G and all γ in Γ.

Let H be a closed subgroup of G, and consider the left coset space G/H. We write $\bar{x} := xH$ for its elements. We endow G/H with a topology, the so-called quotient topology, by calling a subset $O \subset G/H$ open if its inverse image under the natural projection $\pi: x \to \bar{x}$ is open. Then π is continuous by definition, and, since $\pi^{-1}(\pi(S)) = SH$ for any subset $S \subset G$, π is also open. This implies that G/H, being the continuous, open image of a locally compact group, is itself locally compact. It is easily verified that G/H is second countable and Hausdorff. Finally, the natural action of G on G/H, defined by $x\bar{y} := \overline{xy}$, is continuous (since π is open) and transitive. Hence, G/H is a homogeneous space of G. In fact, each homogeneous space of G is G-homeomorphic with a coset space G/H for some closed subgroup H of G.

Indeed, let Γ be a homogeneous space of G, fix a point γ_0 of Γ, and set

$$H := \{x \in G;\ x(\gamma_0) = \gamma_0\}.$$

Then H is a closed subgroup of G, the so-called *stabilizer* (or *little group*) of γ_0. Consider the mapping

$$\beta\colon \bar{x} \in G/H \to x(\gamma_0).$$

Obviously, β is continuous and bijective, and $\beta(y\bar{x}) = y\beta(\bar{x})$ for all $\bar{x}$ in G/H and all y in G. By means of the Baire category theorem (RUDIN [18, 2.2]) we show that β is open. Since the natural mapping $\pi\colon G \to G/H$ is continuous, it suffices to show that $\beta \circ \pi\colon x \in G \to x(\gamma_0)$ is open. For this purpose, we prove that $\beta \circ \pi$ maps any neighbourhood of the identity $e \in G$ onto a neighbourhood of γ_0. Let V be any neighbourhood of e, and choose another open neighbourhood W of e such that (i) $W = W^{-1}$, (ii) $W^2 \subset V$ and (iii) the closure of W is compact (one readily checks that this is possible). Since G is second countable, there exists a countable sequence $x_1, x_2, \ldots$ of elements of G such that

$$G = \bigcup_{i=1}^{\infty} x_i W.$$

Hence, Γ is the union of the countable sequence of compact subsets $\{x_i\bar{W}(\gamma_0)\}_{i=1}^{\infty}$. Since Γ is locally compact and Hausdorff, we can apply the Baire theorem, which asserts that in such a space the countable union of nowhere dense subsets is again nowhere dense, and conclude that for some x_{i_0} the set $x_{i_0}\bar{W}(\gamma_0)$ has a nonvoid interior. Let $x_{i_0}w(\gamma_0)$ be an interior point of $x_{i_0}\bar{W}(\gamma_0)$. Then we have

$$\gamma_0 \in w^{-1}x_{i_0}^{-1}(x_{i_0}W)(\gamma_0) = w^{-1}W(\gamma_0) \subset V(\gamma_0).$$

Consequently, γ_0 is interior to $V(\gamma_0) = (\beta\circ\pi)(V)$, which ends our demonstration (this proof is taken from BOURBAKI [3]).

<u>EXAMPLE 2.2</u>. Consider the unit sphere S^{n-1} in $\mathbb{R}^n$. The special orthogonal group SO(n) acts continuously and transitively on S^{n-1} by rotations. The

stabilizer of the pole $(1,0,\dots,0) \in S^{n-1}$ consists of all matrices

$$\begin{pmatrix} 1 & 0 \ \dots & 0 \\ 0 & & \\ \vdots & R & \\ 0 & & \end{pmatrix} \quad \text{with } R \in SO(n-1),$$

and is therefore naturally isomorphic with $SO(n-1)$. Hence, S^{n-1} is homeomorphic with $SO(n)/SO(n-1)$.

We proceed to state two lemmata which will be used in the next subsection.

LEMMA 2.3. (Urysohn) *Let* X *be a locally compact Hausdorff space, and let* K *and* O *be subsets of* X, *with* K *compact and* O *open, such that* $K \subset O$. *Then there exists a continuous function* f *on* X *with compact support, such that*

(i) $0 \le f(x) \le 1, \quad \forall x \in X;$

(ii) $f(x) = 1 \quad , \quad \forall x \in K;$

(iii) $f(x) = 0 \quad , \quad \forall x \in X \backslash O.$

For a proof we refer to RUDIN [17, 2.12].

LEMMA 2.4. *Let* $K \subset G/H$ *be a compact subset. Then there exists a compact subset* $K' \subset G$ *such that* K' *is mapped onto* K *by the natural mapping* $\pi\colon G \to G/H$.

PROOF. Choose an open neighbourhood U of the identity $e \in G$, such that the closure of U is compact. Then we have

$$K \subset \bigcup_{i=1}^{n} \pi(x_i U),$$

for certain elements $x_1,\dots,x_n$ in G. If we set

$$K' := \left(\bigcup_{i=1}^{n} (x_i \bar{U})\right) \cap \pi^{-1}(K),$$

then K' is compact and $\pi(K') = K$. $\square$

Finally, we state without proof an interesting result, due to Mackey.

By a *Borel cross-section* we will mean a Borel mapping s: $G/H \to G$ which satisfies

$$\pi \circ s = id_{G/H}.$$

LEMMA 2.5. (Mackey) *If* G *is a* lcsc. *group and* H *a closed subgroup of* G, *then there always exists a Borel cross-section* s: $G/H \to G$.

In fact a more general result is true. The proof is based on a classical theorem of Morse and Federer, and can be found in MACKEY [14], or VARADARAJAN [21, thm. 8.11].

It is important to observe that the projection π generally does not admit a continuous cross-section. For instance, set $G = \mathbb{R}$ and $H = \mathbb{Z}$. Then $G/H = \mathbb{T}$, the circle group, and $\pi(x) = e^{2\pi ix}$. It can be shown easily that no mapping $s: \mathbb{T} \to \mathbb{R}$ exists which is continuous and satisfies $\pi \circ s = id_{\mathbb{T}}$.

2.3. Quasi-invariant measures on coset spaces

Throughout this subsection, all measures will be assumed to be positive nonzero Borel measures. Let G be a locally compact second countable (lcsc.) group, H a closed subgroup of G, and consider the homogeneous space G/H. Elements of this space are denoted by $\bar{x}$, where $\pi: x \to \pi(x) = \bar{x}$ is the natural projection of G onto G/H. For $S \subset G/H$ and $x \in G$ we write $x[S] = \{x\bar{y}; \bar{y} \in S\}$.

A measure μ on G/H is said to be G-invariant (shortly: invariant) if $\mu = \mu_x$ for all x in G. Here μ_x denotes the translated measure, defined by $\mu_x(B) := \mu(x[B])$, for Borel sets B in G/H. Thus, μ is invariant if and only if

$$(2.1) \qquad \mu(B) = \mu(x[B]), \qquad \forall x \in G, \forall B \in \mathcal{B}(G/H).$$

(We write $\mathcal{B}(X)$ for the collection of all Borel subsets of a Borel space X).

For instance, if H is invariant in G, then the space G/H becomes a lcsc. group in its own right, with respect to the quotient topology, if we define a product by $\bar{x}\bar{y} := \overline{xy}$. Since $x\bar{y} = \overline{xy}$ for all x,y in G, we see that the left Haar measure on G/H satisfies (2.1). Hence, in this case an invariant measure always exists, and, moreover, it is unique up to constant factor.

Returning to the general case, let ν be a left Haar measure on G, and set

$$\mu(B) := \nu(\pi^{-1}(B)), \qquad B \in \mathcal{B}(G/H).$$

Then μ is a positive σ-additive function on $\mathcal{B}(G/H)$, and $\mu(\emptyset) = 0$. Hence, μ is a measure in the ordinary sense on the Borel subsets of G/H, and, since $\pi^{-1}(x[B]) = x\pi^{-1}(B)$ for all x in G and all B in $\mathcal{B}(G/H)$, it satisfies (2.1). However, if C is a compact subset of G/H, then $\pi^{-1}(C)$ is not necessarily compact in G, and $\nu(\pi^{-1}(C))$ can be infinite (and it will be, in certain cases). Hence, μ fails in general to be finite on compact sets, which is a requirement for Borel measures. Notice that if H is compact, $\pi^{-1}(C)$ is compact for each compact subset C of G/H. Consequently, μ is a G-invariant measure in this case. The intention of this discussion is to make plausible that what would seem a natural way to obtain invariant measures on coset spaces does not work in general. As we will show later on in this subsection, there are homogeneous spaces on which no invariant measure exists at all. Therefore, we will focus on measures with a weaker invariance property than (2.1). Recall that a measure μ is said to be absolutely continuous with respect to another measure ν on the same space, if each null-set for ν is also a null-set for μ; notation: $\mu << \nu$. Two measures μ and ν on the same space are called equivalent (notation $\mu \simeq \nu$) if $\mu << \nu$ and $\nu << \mu$.

DEFINITION 2.6. A measure μ on the coset space G/H is called *quasi-invariant* if it is equivalent to each of its translates, i.e. $\mu \simeq \mu_x$ for all x in G.

The classes of measures corresponding to the equivalence relation $\simeq$, are called measure classes, and such a class is denoted by $[\mu]$, where μ is a representative. A measure class $[\mu]$ on G/H is called invariant if

$$\mu' \in [\mu] \Rightarrow \mu'_x \in [\mu], \qquad \forall x \in G.$$

We can now restate the above definition as follows: A measure μ on G/H is called quasi-invariant if it belongs to an invariant measure class (notice that $\mu \simeq \mu' \Rightarrow \mu_x \simeq \mu'_x$).

We may give still another characterization of quasi-invariant measures, by utilizing the well-known Radon-Nykodym theorem, which gives a necessary and sufficient condition for two measures to be equivalent. Indeed, call a measure μ on G/H quasi-invariant if and only if for each y in G there exists a strictly positive Borel function $\bar{x} \to R(\bar{x},y)$ on G/H such that

$$(2.2) \qquad \int_{G/H} f(y^{-1}\bar{x})d\mu(\bar{x}) = \int_{G/H} f(\bar{x})R(\bar{x},y)d\mu(\bar{x}),$$

for all f in K (G/H), the space of continuous complex-valued functions on G/H with compact support.

In this subsection we will prove that there always exists a unique invariant measure class on G/H. Moreover, we will show that this class always contains a measure μ for which the function R occuring in (2.2) can be taken to be continuous in both variables (considered as a function on G/H $\times$ G). As a corollary of a certain stage of the existence proof we will obtain a necessary and sufficient condition for the existence of an invariant measure on G/H.

We start with the discussion of a very useful relationship between the spaces K(G) and K(G/H). We fix Haar measures ν_G and ν_H on G and H, respectively. If f belongs to K(G), then consider the expression

$$\int_H f(xh)d\nu_H(h), \qquad x \in G.$$

The value of this integral remains constant if we let x run through a left H-coset. Hence, if we set

$$\tilde{f}(\bar{x}) := \int_H f(xh)d\nu_H(h),$$

then we obtain a function $\tilde{f}$ on the coset space G/H.

LEMMA 2.7. *The assignment* $f \to \tilde{f}$ *maps* K(G) *onto* K(G/H). *Furthermore,* $f \geq 0$ *implies* $\tilde{f} \geq 0$.

PROOF. Let $f \in K(G)$. Clearly, the support of $\tilde{f}$ is contained in $\pi(\mathrm{supp}(f))$. Continuity of $\tilde{f}$ can be verified by simple standard arguments, exploring the fact that f is uniformly continuous. Hence, $\tilde{f} \in K(G/H)$.

Next, let $g_1 \in K(G/H)$, and set $K = \text{supp}(g_1)$. Then we can choose a compact subset K' of G such that $\pi(K') = K$ (lemma 2.4). There exists a positive function $g_2 \in K(G)$ with $g_2(x) = 1$ for all $x \in K'$ (lemma 2.3). If $x \in \pi^{-1}(K)$, then there exists an element $h \in H$ with $xh \in K'$. Hence, $\tilde{g}_2(\bar{x}) > 0$ for all $\bar{x} \in K$. Define a function f on G by

$$f(x) := \begin{cases} \dfrac{g_1(\bar{x})g_2(x)}{\tilde{g}_2(x)} & \text{if } x \in \pi^{-1}(K), \\ \\ 0 & \text{otherwise.} \end{cases}$$

Clearly f is compactly supported, and from the fact that $K = \text{supp}(g_1)$ and the continuity of g_1, g_2 and $\tilde{g}_2$ it follows that f is continuous. Furthermore,

$$\tilde{f}(\bar{x}) = \frac{g_1(\bar{x})}{\tilde{g}_2(\bar{x})} \int_H g_2(xh)\,d\nu_H(h) = g_1(\bar{x}),$$

so $\tilde{f} = g$. The second assertion of the lemma is obvious. □

Let μ be a measure on G/H. Then, for each $f \in K(G)$, we define $\mu^{\#}(f) := \mu(\tilde{f})$. From the preceding lemma and the obvious linearity of $f \to \tilde{f}$ it follows that $\mu^{\#}$ defines a measure on G, uniquely determined by μ. Hence, we have obtained a mapping $\mu \to \mu^{\#}$ from the set of measures on G/H into the set of measures on G. [This mapping can be considered as the dual of $f \to \tilde{f}$]. Before we state the properties of this important mapping, we render the following useful extension of lemma 2.7.

LEMMA 2.8. *The mapping* $f \to \tilde{f}$ *on* $K(G)$ *extends to the space of Borel functions* f *on* G *which satisfy*

$$\int_H |f(xh)|\,d\nu_H(h) < \infty, \qquad \forall x \in G.$$

Moreover, for such functions one has

(i) f *is a Borel function on* G/H;
(ii) $f \geq 0$ *implies* $\tilde{f} \geq 0$ *and* $\mu^{\#}(f) = \mu(\tilde{f})$ *for any measure* μ *on* G/H.

PROOF. By virtue of the Tonelli theorem, the function

$$x \to \int_H f(xh)d\nu_H(h) \qquad (x \in G)$$

is a Borel function on G. Being invariant on left cosets, it defines a unique Borel function $\tilde{f}$ on G/H. Clearly, $f \geq 0$ implies $\tilde{f} \geq 0$. The second implication in (ii) is verified by applying some standard arguments from elementary measure theory to lemma 2.7 (cf. VARADARAJAN [21], lemma 8.15]). Note that, since f is compactly supported and bounded, it follows that $\tilde{f}$ also has these properties (ibidem). $\square$

COROLLARY 2.9. *For any measure* μ *on* G/H *one has for each Borel set* B *in* G/H: $\mu(B) = 0$ *iff* $\mu^{\#}(\pi^{-1}(B)) = 0$.

PROOF. Let B be a Borel set in G/H. First, suppose $\mu(B) = 0$, and let K be a compact subset of G with $K \subset \pi^{-1}(B)$. Then, by lemma 1.8,

$$\mu^{\#}(K) = \int_{G/H} \tilde{\chi}_K(\bar{x})d\mu(\bar{x}).$$

Since $\tilde{\chi}_K$ vanishes outside B, this yields $\mu^{\#}(K) = 0$. But then, by virtue of regularity, $\mu^{\#}(\pi^{-1}(B)) = 0$. Conversely, suppose $\mu^{\#}(\pi^{-1}(B)) = 0$, and let K be a compact subset of G/H contained in B. There exists an increasing sequence of compact subsets $K_1, K_2, \ldots$ of G with $\pi^{-1}(K) = \bigcup_{n=1}^{\infty} K_n$. From the preceding lemma it follows that $\tilde{\chi}_{K_n} = 0$ a.e. $[\mu]$ for all n. Now, suppose $\mu(K) > 0$. Then, for some $\bar{x} \in G/H$, we must have $\tilde{\chi}_{K_n}(\bar{x}) = 0$ for all n. Since

$$\tilde{\chi}_{K_n}(\bar{x}) = \int_H \chi_{K_n}(xh)d\nu_H(h),$$

the sets $x^{-1}K_n \cap H$ must be ν_H - null sets. This contradicts the obvious fact that $x^{-1}K_n \cap H \uparrow H$. $\square$.

COROLLARY 2.10. *Let μ_1 and μ_2 be measures on G/H. Then $\mu_1 << \mu_2$ if and only if $\mu_1^{\#} << \mu_2^{\#}$. Furthermore, if $\mu_1 << \mu_2$, then*

$$\frac{d\mu_1}{d\mu_2}(\bar{x}) = \frac{d\mu_1^{\#}}{d\mu_2^{\#}}(\bar{x}), \qquad x \in G.$$

PROOF. The "if" part of the first statement immediately follows from corollary 2.9. As to the other assertions, let μ_1 and μ_2 be measures on G/H with $\mu_1 << \mu_2$. By virtue of the Radon-Nikodym theorem there exists a positive Borel function ϕ on G/H such that

$$\mu_1(f) = \int_{G/H} f(\bar{x})\phi(\bar{x})d\mu_2(\bar{x}),$$

for all Borel functions f on G/H. If $g \in K(G)$ then one readily verifies that $(g(\phi\circ\pi))^{\sim} = \tilde{g}\phi$. But then, by lemma 2.8, it follows that

$$\int_G g(x)\phi(\bar{x})d\mu_2^{\#}(x) = \int_{G/H} \tilde{g}(\bar{x})\phi(\bar{x})d\mu_2(\bar{x})$$

$$= \int_{G/H} \tilde{g}(\bar{x})d\mu_1(\bar{x}).$$

$$= \int_G g(x)d\mu_1^{\#}(x).$$

Hence $\mu_1^{\#} << \mu_2^{\#}$, and, in particular, the Radon-Nikodym derivative $d\mu_1^{\#}/d\mu_2^{\#}$ equals $\phi \circ \pi$. □

THEOREM 2.11. *Let μ, μ_1, μ_2 be measures on G/H. Then* (i) $\mu_1 \simeq \mu_2$ *if and only if* $\mu_1^{\#} \simeq \mu_2^{\#}$, *and* (ii) μ *is (quasi-)invariant if and only if* $\mu^{\#}$ *is (quasi-) invariant.*

(Note that definition 2.6 also defines quasi-invariant measures on G.)

PROOF. The first statement follows immediately from corollary 2.10. The second one follows from the first one and from the obvious observation

$$(\mu_x)^{\#} = (\mu^{\#})_x, \qquad \forall x \in G. \quad \square$$

The following lemma, in combination with the statements of theorem 2.11, establishes the uniqueness of an invariant measure class on G/H (if it exists).

LEMMA 2.12. *Each quasi-invariant measure on* G *is equivalent to the Haar measures on* G.

PROOF. Let μ be a quasi-invariant measure on G, and let $B \in \mathcal{B}(G)$. Then we have

$$\int_G \int_G \chi_{B^{-1}}(x)\,d\nu_G(x)\,d\mu(y) =$$

$$= \int_G \int_G \chi_{B^{-1}}(y^{-1}x)\,d\nu_G(x)\,d\mu(y)$$

$$= \int_G \int_G \chi_{B^{-1}}(y^{-1}x)\,d\mu(y)\,d\nu_G(x)$$

$$= \int_G \int_G \chi_{x[B]}(y)\,d\mu(y)\,d\nu_G(x)$$

$$= \int_G \mu(x[B])\,d\nu_G(x).$$

Elementary considerations show that these steps are all legitimate. Now, if $\mu(B) = 0$, then $\mu(x[B]) = 0$, and hence $\nu_G(B^{-1}) = 0$. But B^{-1} has Haar measure zero if and only if B has Haar measure zero. Hence, $\nu_G(B) = 0$. Clearly this argument can be reversed; so, $\mu \in [\nu_G]$. □

If we can show that the image of the mapping $\mu \to \mu^{\#}$ contains a quasi-invariant measure, then the existence of an invariant measure class on G/H follows at once from theorem 2.11 (ii). For this purpose, we first determine this image.

Let Δ_H and Δ_G denote the Haar moduli of H and G respectively.

LEMMA 2.13. *Let* ν *be a measure on* G. *Then there exists a measure* μ *on* G/H *with* $\nu = \mu^{\#}$ *if and only if*

$$\int_G f(xh)\,d\nu(x) = \Delta_H(h^{-1}) \int_G f(x)\,d\nu(x), \qquad \forall f \in K(G),\ \forall h \in H. \tag{2.3}$$

PROOF. Suppose that ν is equal to $\mu^{\#}$, for a certain measure μ on G/H. Then $\nu(f) = \mu(\tilde{f})$ for all f in $K(G)$, which yields for h_0 fixed in H:

$$\begin{aligned}
\int_G f(xh_0)\,d\nu(x) &= \int_G f_{h_0}(x)\,d\nu(x) \\
&= \int_{G/H}\int_H f_{h_0}(xh)\,d\nu_H(h)\,d\mu(\bar{x}) \\
&= \int_{G/H}\int_H \Delta_H(h_0^{-1}) f(xh)\,d\nu_H(h)\,d\mu(\bar{x}) \\
&= \Delta_H(h_0^{-1}) \int_G f(x)\,d\nu(x).
\end{aligned}$$

(Here we use f_h to denote the function $x \to f(xh)$).

Next, let ν be a measure on G which satisfies (2.3). Then, for $\tilde{f} \in K(G/H)$, we set $\mu(\tilde{f}) := \nu(f)$. We first show that this definition is legitimate, by proving that $\tilde{f}_1 = \tilde{f}_2$ implies $\nu(f_1) = \nu(f_2)$. This property of ν follows primarily from (2.3).

Let f belong to $K(G)$. By virtue of the lemmata 2.3 and 2.7, we can choose a function g in $K(G)$ such that $\tilde{g}(\bar{x}) = 1$ for all $\bar{x}$ in $\pi(\mathrm{supp}(f))$. Utilizing formula (2.3) and applying the Fubini theorem, we can make the following computation:

$$\begin{aligned}
\int_G f(x)\,d\nu(x) &= \int_G f(x) \left(\int_H g(xh)\,d\nu_H(h) \right) d\nu(x) \\
&= \int_H \int_G f(x) g(xh)\,d\nu(x)\,d\nu_H(h) \\
&= \int_H \Delta_H(h^{-1}) \left(\int_G f(xh^{-1}) g(x)\,d\nu(x) \right) d\nu_H(h)
\end{aligned}$$

$$= \int_G g(x) \left(\int_H \Delta_H(h^{-1}) f(xh^{-1}) d\nu_H(h) \right) d\nu(x)$$

$$= \int_G g(x) \left(\int_H f(xh) d\nu_H(h) \right) d\nu(x)$$

$$= \int_G g(x) \tilde{f}(\bar{x}) d\nu(x).$$

But then, if $\tilde{f}(\bar{x}) = 0$ for all $\bar{x}$ in G/H, we have $\nu(f) = 0$. By linearity of the mapping $f \to \tilde{f}$ it follows that the number $\mu(\tilde{f}) = \nu(f)$ is well-defined.

Clearly, μ is a linear functional. Furthermore, by means of the proof of lemma 2.7, it can be easily verified that for each $g \in K(G/H)$ with $g \geq 0$, a function $f \in K(G)$ can be chosen such that $f \geq 0$ and $\tilde{f} = g$. This shows that μ is positive. Now it follows from the Riesz representation theorem (RUDIN [17]) that μ is a measure. This finishes our proof, since $\mu^{\#} = \nu$ by definition. □

If we set $\nu = \nu_G$, the Haar measure on G, then the identity (2.3) reduces to

$$d\nu_G(x) = \frac{\Delta_G(h)}{\Delta_H(h)} d\nu_G(x), \qquad \forall h \in H.$$

Clearly, this will only be true if Δ_G restricted to H is equal to Δ_H. From theorem 2.11 (ii) it follows that G/H admits an invariant measure if and only if the Haar measure on G lies in the image of the mapping $\mu \to \mu^{\#}$. Hence, lemma 2.13 yields the following criterion for the existence of an invariant measure on G/H:

<u>COROLLARY 2.14</u>. *The coset space* G/H *admits an invariant measure if and only if*

$$\Delta_G(h) = \Delta_H(h), \qquad \forall h \in H. \qquad \square$$

Next we turn to the problem of proving the existence of quasi-invariant measures on G/H. This problem is reduced by lemma 2.13 to the problem of finding quasi-invariant measures on G which satisfy (2.3). As we will demonstrate below, this problem is solved in a very nice way by the following crucial lemma:

LEMMA 2.15. *There exist continuous, strictly positive solutions of the functional equation*

$$(2.4) \qquad \rho(x) = \frac{\Delta_G(h)}{\Delta_H(h)}\,\rho(xh), \qquad \forall x \in G, \forall h \in H.$$

The proof of this lemma is rather technical, and we will give it in an appendix (page 50). Define a measure ν on G by

$$(2.5) \qquad d\nu(x) = \rho(x)\,d\nu_G(x), \qquad x \in G,$$

where ρ is a continuous, strictly positive solution of (2.4). Then

$$\begin{aligned}\int_G f(xh)\,d\nu(x) &= \Delta_G(h^{-1}) \int_G f(x)\rho(xh^{-1})\,d\nu_G(x)\\ &= \Delta_H(h^{-1}) \int_G f(x)\,d\nu(x), \qquad f \in K(G),\end{aligned}$$

so there exists a measure μ on G/H with $\mu^{\#} = \nu$. Furthermore, we have

$$\begin{aligned}\int_G f(y^{-1})\,d\nu(x) &= \int_G f(x)\rho(yx)\,d\nu_G(x)\\ &= \int_G f(x)\,\frac{\rho(yx)}{\rho(x)}\,d\nu(x), \qquad f \in K(G).\end{aligned}$$

Comparing this to the characterization of quasi-invariant measures we gave by means of the Radon-Nikodym theorem, we can conclude that ν is quasi-invariant, which solves our problem.

Furthermore, note that the Radon-Nikodym derivative $d\nu_y(x)/d\nu(x)$ is given by $\rho(yx)/\rho(x)$. By virtue of lemma 2.10, we have

$$\frac{d\mu_1}{d\mu_2}(\bar{x}) = \frac{d\mu_1{}^{\#}}{d\mu_2{}^{\#}}(x),$$

if μ_1 and μ_2 are equivalent measures on G/H. Hence, for $\mu^{\#} = \mu$, we have

$$\frac{d\mu_y}{d\mu}(\bar{x}) = \frac{d\nu_y}{d\nu}(x) = \frac{\rho(yx)}{\rho(x)} ,$$

so

$$(2.2) \qquad \int_{G/H} f(y^{-1}\bar{x})\,d\mu(\bar{x}) = \int_{G/H} f(\bar{x})R(\bar{x},y)\,d\mu(\bar{x}), \quad f \in K(G/H),$$

with

$$R(\bar{x},y) = \frac{\rho(yx)}{\rho(x)} .$$

From the last identity we infer the fact that $R(-,-)$, considered as a function on $G/H \times G$ is continuous in both variables. We emphasize that in this case $R(-,-)$ is uniquely determined by μ.

For future reference we state some useful properties of continuous R-functions corresponding to quasi-invariant measures on G/H by formula (2.2). They can be verified by direct computation.

(2.6) (i) $\quad R(\bar{x},yz) = R(z\bar{x},y)R(\bar{x},z), \quad \bar{x} \in G/H,\ y,z \in G;$

(2.6) (ii) $\quad R(\bar{x},e) = 1 \quad , \quad \bar{x} \in G/H;$

(2.6) (iii) $\quad (R(\bar{x},y))^{-1} = R(y\bar{x},y^{-1}) \quad , \quad \bar{x} \in G/H, y \in G.$

(2.6) (iv) $\quad R(\bar{e},h) = \dfrac{\Delta_H(h)}{\Delta_G(h)} \quad , \quad h \in H.$

Suppose that we are given a continuous strictly positive function $R: G/H \times G \to \mathbb{R}^{>0}$, which satisfies (2.6)(i) and (2.6)(iv). Then, if we set

$$\rho(y) = R(\bar{e},y), \qquad y \in G,$$

we find

$$\rho(yh) = R(\bar{e},yh) = R(\bar{e},y).R(\bar{e},h)$$

$$= \rho(y).\frac{\Delta_H(h)}{\Delta_G(h)} , \qquad y \in G,\ h \in H.$$

In this way we obtain a quasi-invariant measure corresponding to R.

REMARK. Let ρ_1 and ρ_2 be two continuous strictly positive solutions of (2.4), and let μ_1 and μ_2 be the two corresponding quasi-invariant measures on G/H. Then, by virtue of corollary 2.10 and formula (2.5), we have

$$\frac{d\mu_1}{d\mu_2}(\bar{x}) = \frac{\rho_1(x)}{\rho_2(x)} .$$

EXAMPLE 2.16. In example 2.2 we showed that the homogeneous space $SO(n)/SO(n-1)$ is homeomorphic with the unit sphere S^{n-1} of dimension n-1. Since $SO(n-1)$ is a compact subgroup of $SO(n)$ for all $n = 1,2,\ldots$, there exists an invariant measure on S^{n-1}. This is the well-known rotation invariant measure.

EXAMPLE 2.17. Consider the subgroup H of $G\ell(2,\mathbb{R})$ consisting of all real matrices

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \quad \text{with } a > 0.$$

The group $G\ell(2,\mathbb{R})$ can be identified in a natural way with a subset of $\mathbb{R}^4$. Let λ^4 be the Lebesgue measure on $\mathbb{R}^4$ and set

$$d\nu_{G\ell(2,\mathbb{R})}(x) := \frac{d\lambda^4(x)}{|\det(x)|^2}, \qquad x \in G\ell(2,\mathbb{R}).$$

Then one readily verifies that ν is a left and right invariant measure on $G\ell(2,\mathbb{R})$, and therefore this group is unimodular. However, if we let λ denote the Lebesgue measure on $\mathbb{R}$, and if we set

$$d\nu_H\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right) := \frac{d\lambda(a)d\lambda(b)}{a^2},$$

then ν_H defines a left Haar measure on H, which is obviously not right invariant. The modular function on H can be found by solving the equation

$$d\nu_H(xy) = \Delta_H(y)d\nu_H(x), \qquad x,y \in H.$$

This yields

$$\Delta_H\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right) = a^{-1} = \det\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{-1}\right) .$$

Define a function ρ on $G\ell(2,\mathbb{R})$ by

$$\rho(x) := |\det(x)|^{-1}.$$

Then ρ is a strictly positive continuous solution of equation (2.4). Hence, the measure ν on $G\ell(2,\mathbb{R})$ defined by

$$d\nu(x) := \rho(x)d\nu_{G\ell(2,\mathbb{R})}(x)$$

is quasi-invariant on $G\ell(2,\mathbb{R})$ and lies in the image of $\mu \longrightarrow \mu^{\#}$. The quasi-invariant measure μ on $G\ell(2,\mathbb{R})/H$ with $\mu^{\#} = \nu$ can now be expressed in terms of $\nu_{G\ell(2,\mathbb{R})}$ and ρ. The corresponding R-function on $G\ell(2,\mathbb{R})/H \times$ $\times\ G\ell(2,\mathbb{R})$ is given by

$$R(\bar{x},y) = \frac{\rho(yx)}{\rho(x)} = |\det\ (y)|^{-1}$$

Notice that this function is independent of $\bar{x}$. This means that the Radon-Nikodym derivative $d\mu_y/d\mu$ is constant for all y in $G\ell(2,\mathbb{R})$. Quasi-invariant measures with this property are called relatively invariant. One easily proofs the following criterion for the existence of relatively invariant measures on a coset space G/H:

<u>THEOREM 2.18</u>. *There exist relatively invariant measures on* G/H *if and only if the function* ρ *of lemma* 2.15 *can be chosen such that* $\rho(x)\rho(y) = \rho(xy)$ *for all* x,y *in* G.

<u>EXAMPLE 2.19</u>. Consider the case where G is the product of two closed subgroups K and H, with $K \cap H = \{e\}$. We will derive a rather simple way to find an explicit expression for a quasi-invariant measure on G/H, in this case, under the additional assumption that the mapping $kh \longrightarrow (k,h)$ from G onto $K \times H$ be continuous. Then G and $K \times H$ are homeomorphic. This implies

$$G/H \approx (K\times H)/H \approx K,$$

where the homeomorphism from G/H onto K is given by sending $\bar{x} = xH$ to the projection of x on K. We denote the projection of G on K and H by π_1 and π_2, respectively, that is,

$$\pi_1(kh) := k, \quad \pi_2(kh) := h, \qquad k \in K,\ h \in H.$$

Define a function ρ on G by

$$\rho(x) := \frac{\Delta_H(\pi_2(x))}{\Delta_G(\pi_2(x))} .$$

Then ρ is single-valued, continuous and strictly positive. Moreover, it satisfies (2.4). Denote by μ the corresponding quasi-invariant measure on G/H. For the R-function we find

$$R(\bar{x},y) = \frac{\Delta_H(\pi_2(yx)(\pi_2(x))^{-1})}{\Delta_G(\pi_2(yx)(\pi_2(x))^{-1})} .$$

If we identify the homeomorphic spaces G/H and K, this expression reduces to

$$R(k,y) = \frac{\Delta_H(\pi_2(yk))}{\Delta_G(\pi_2(yk))} .$$

In particular

$$R(k,y) = 1, \qquad y \in K.$$

Therefore, μ is invariant for the G-action on G/H restricted to K, so μ is, under the above identification, equal to the left Haar measure on K. (For, the K-action on G/H reduces to left multiplication under this identification).

For instance, the situation sketched above is encountered in the case of semi-simple Lie groups which are non-compact and connected. Indeed, these groups admit a so-called Iwasawa decomposition G = KAN, where K is compact, A is abelian and closed, and N is nilpotent and closed. Moreover, it is known that the mapping $(k,a,n) \longrightarrow kan$ from $K \times A \times N$ onto G is an analytic diffeomorphism (see HELGASON [7,thm. VI 5.1]). If we set H = AN, we obtain the situation above.

Consider for example the case $G = G\ell(2,\mathbb{R})$, the group of real 2×2 matrices with determinant 1. Then $K = SO(2)$, the special orthogonal group in two dimensions, and

$$A = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix};\ a > 0\right\},\ N = \left\{\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix};\ c \in \mathbb{R}\right\}.$$

Hence

$$H = \left\{\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix};\ a > 0,\ c \in \mathbb{R}\right\}.$$

It is rather tedious to compute explicit expressions for π_1 and π_2 in this case, and therefore we use another method. The group $S\ell(2,\mathbb{R})$ acts on the one-dimensional real projective space $\mathbb{P}_1(\mathbb{R})$. This space can be obtained by identifying points $\neq 0$ of $\mathbb{R}^2$ which are scalar multiples of each other. By choosing so-called inhomogeneous coordinates, we can identify $\mathbb{P}_1(\mathbb{R})$ with the extended real line $\mathbb{R} \cup \{\infty\}$. Indeed, let $[x,y]$ denote an equivalence class in $\mathbb{R}^2$, and set $[x,y] \to t = \frac{x}{y}$, $y > 0$ and $[x,0] \to \{\infty\}$. The corresponding action of $S\ell(2,\mathbb{R})$ on $\mathbb{R} \cup \{\infty\}$ reads

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(t) = \frac{at+b}{ct+d}.$$

The expression on the right-hand side becomes $\frac{a}{c}$ if $t = \{\infty\}$, and $\{\infty\}$ if $t = -\frac{d}{c}$. The stabilizer of $t = 0$ consists of all real matrices

$$\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix},\quad a \neq 0,$$

and is thus equal to $H \times \mathbb{Z}_2$. One verifies easily that

$$d\mu(t) = \frac{d\lambda(t)}{1+t^2},\quad t \neq \{\infty\},\ \mu(\{\infty\}) = 0,$$

defines an $SO(2)$-invariant measure on $\mathbb{R} \cup \{\infty\}$. Hence, μ is quasi-invariant for the action of $S\ell(2,\mathbb{R})$. We can compute the corresponding R-function directly:

$$d\mu\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}.t\right) = \frac{1}{1+(at+b)^2(ct+d)^{-2}}\, d\lambda\left(\frac{at+b}{ct+d}\right)$$

$$= \frac{a(ct+d)d\lambda(t)-c(at+b)d\lambda(t)}{(at+b)^2+(ct+d)^2} = \frac{(ad-cb)d\lambda(t)}{(at+b)^2+(ct+d)^2}$$

$$= \frac{d\lambda(t)}{(at+b)^2+(ct+d)^2} = \frac{1+t^2}{(at+b)^2+(ct+d)^2}\, d\mu(t)$$

$$= R\left(t, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) d\mu(t).$$

2.4. Induction for lcsc. groups

Let G be a lcsc. group, and let τ be a unitary representation of a closed subgroup H of G. Define a linear space $F_\tau(G,H(\tau))$ to consist of all $H(\tau)$ - valued functions f on G, which satisfy

$$(2.7) \qquad f(xh) = \tau(h^{-1})f(x), \qquad \forall x \in G, \forall h \in H.$$

Analogous to the finite case (§1.4) we define for each x in G a linear mapping $\hat{\tau}(x)$ from F_τ onto itself, by

$$(\hat{\tau}(x)f)(y) := f(x^{-1}y), \qquad f \in F_\tau.$$

Then $\hat{\tau}$ is a homomorphism from G into the space of linear operators on F_τ. In general, the space F_τ is "too big" for $\hat{\tau}$ to be a representation of G in the sense of §2.1. Therefore, the obvious thing to do, will be to search for a $\hat{\tau}$-invariant linear subspace, on which an inner product can be defined, which is respected by $\hat{\tau}$. Then $\hat{\tau}$ will be a proper unitary representation of G if it is extended to the completion of this subspace and if we can prove continuity of $\hat{\tau}$.

Consider first the case where the coset space G/H admits an invariant measure, say μ. Let K_τ be the linear subspace of F_τ consisting of all functions f in F_τ which are continuous and have compact support "modulo H", i.e. supp(f) $\subset$ K.H for some compact subset K of G. It is clear at once that this is a $\hat{\tau}$-invariant subspace. For f_1, $f_2 \in K_\tau$, the complex-valued function $x \longrightarrow (f_1(x),f_2(x))$ on G (inner product in $H(\tau)$), is continuous and has compact support modulo H. Moreover, by virtue of (2.7) and the fact that τ is unitary, it is also constant on left H-cosets in G. Consequently, we obtain a well-defined continuous function on G/H by setting $\bar{x} \longrightarrow (f_1(x),f_2(x))$, which is readily seen to be compactly supported in G/H. Led by these considerations, we define a positive definite bilinear form on K_τ by

$$(2.8) \qquad (f_1,f_2) := \int_{G/H} (f_1(x),f_2(x))d\mu(\bar{x}).$$

Since μ is G-invariant, it follows from the definition of $\hat{\tau}$ that this inner product is $\hat{\tau}$-invariant. Let y, $y_0 \in G$, then

$$|(\hat{\tau}(y)f_1,f_2)-(\hat{\tau}(y_0)f_1,\ f_2)| \leq$$

$$\leq \int_{G/H} (f_1(y^{-1}x),f_2(x))-(f(y_0^{-1}x),f_2(x))|d\mu(\bar{x})$$

$$\leq \int_{G/H} \|f_1(y^{-1}x)-f_1(y_0^{-1}x)\|^2\ \|f_2(x)\|^2 d\mu(\bar{x}).$$

By an elementary computation one verifies that for each f in K_τ, the function $x \longrightarrow \|f(x)\|$ on G is uniformly continuous. Hence, we can choose a compact neighbourhood K_e of the identity e in G with

$$yy_0^{-1} \in K_e \Rightarrow \|f_1(y^{-1}x)-f_1(y_0^{-1}x)\| < \varepsilon,$$

which implies

$$yy_0^{-1} \in K_e \Rightarrow |(\hat{\tau}(y)f_1,f_2)-(\hat{\tau}(y_0)f_1,f_2)| < \varepsilon.C,$$

where C is a constant, independent of K_e. This proves weak continuity of $\hat{\tau}$ restricted to K_τ, and since $\hat{\tau}$ is unitary on this space, strong continuity follows at once (lemma 2.1). Hence we have demonstrated that $\hat{\tau}$ can be extended to a unitary representation of G on the completion $\bar{K}_\tau$ of K_τ. We shall call this representation induced by τ, and denote it by τ^G. Furthermore, we write $\bar{K}_\tau =: H(\tau^G)$.

Next, consider the case where G/H only admits a quasi-invariant measure. Let μ be a quasi-invariant measure on G/H, such that the corresponding R-function is continuous. (Recall that R is a strictly positive function on G/H × G defined by $R(\bar{x},y) = (d\mu_y/d\mu)(\bar{x})$.) If we consider again the space

$K_\tau \subset F_\tau$, with the inner product defined by (2.8), this time for the quasi-invariant measure μ, then

$$\|\hat{\tau}(y)f\|^2 = \int_{G/H} \|f(y^{-1}x)\|^2 \, d\mu(\bar{x})$$

$$= \int_{G/H} \|f(x)\|^2 \, R(\bar{x},y)d\mu(\bar{x}).$$

Hence, $\hat{\tau}$ need not be unitary. However, this identity suggests the following alternative to (2.7):

$$(2.9) \qquad (\hat{\tau}(y)f)(x) := f(y^{-1}x)(R(\bar{x},y^{-1}))^{\frac{1}{2}}, \quad f \in F_\tau.$$

Formula (2.6)(i) ensures that $\hat{\tau}$ is still a homomorphism when defined in this way. Furthermore, K_τ is again an invariant subspace, and $\hat{\tau}$ acts on this space by unitary operators. Weak continuity of $\hat{\tau}$ can be verified in the same way as above. Hence $\hat{\tau}$ can be regarded as a unitary representation of G on the Hilbert space $\bar{K}_\tau$, induced by τ, and we will denote it by τ^G.

After this definition, two questions immediately arise:

(i) Are there nonzero functions in K_τ?

(ii) Does the induced representation τ^G depend on the particular choice of a quasi-invariant measure?

We will answer those questions in the next two lemmata. Let $K(G,H(\tau))$ denote the space of continuous functions on G with range in $H(\tau)$ and compact support. Let f be such a function and consider a new function $\bar{f}$ on G, defined by

$$\bar{f}(x) := \int_H \tau(h)f(xh)d\nu_H(h).$$

Notice that this definition is legitimate, thanks to the continuity properties of τ as an operator-valued function on H.

<u>LEMMA 2.20</u>. *The mapping* $f \longrightarrow \bar{f}$ *is a surjection from* $K(G,H(\tau))$ *onto* K_τ.

PROOF. Let f belong to $K(G,H(\tau))$. Then f is uniformly continuous, and

$$\|\bar{f}(x)-\bar{f}(x_0)\| \leq \int_H \|f(xh)-f(x_0h)\| d\nu_H(h).$$

Hence, continuity of $\bar{f}$ can be proved easily. Furthermore, we have

$$\operatorname{supp}(\bar{f}) \subset \operatorname{supp}(f).H,$$

so $\bar{f}$ has compact support modulo H. Finally,

$$\bar{f}(xh_0) = \int_H \tau(h)f(xh_0h)d\nu_H(h) = \int_H \tau(h_0^{-1}h)f(xh)d\nu_H(h)$$
$$= \tau(h_0^{-1})\bar{f}(x),$$

by virtue of a well-known property of vector-valued integrals. Consequently, $\bar{f}$ belongs to K_τ. We now show surjectivity. Let p be an element of K_τ. By virtue of the lemmata 2.3 and 2.7, we may choose a function $f_1 \in K(G)$ such that $\tilde{f}_1(\bar{x}) = 1$ for all $\bar{x}$ in $\pi(\operatorname{supp}(p))$. (Recall that $\tilde{f}_1 \in K(G/H)$ was defined by

$$\tilde{f}_1(\bar{x}) = \int_H f_1(xh)d\nu_H(h).)$$

Set $f := f_1.p$. Then

$$\bar{f}(x) = \int_H \tau(h)f_1(xh)p(xh)d\nu_H(h) = \int_H f_1(xh)p(x)d\nu_H(h)$$
$$= \tilde{f}_1(\bar{x})p(x) = p(x). \quad \square$$

COROLLARY 2.21. *For each* x *in* G *the subspace*

$$\{f(x) : f \in K_\tau\}$$

of $H(\tau)$ *lies dense in* $H(\tau)$.

PROOF. Suppose that for some x in G and some nonzero ξ in $H(\tau)$ we have

$$(f(x),\xi) = 0, \quad \forall f \in K_\tau.$$

By virtue of lemma 2.20 this amounts to

$$\int_H (\tau(h)g(xh),\xi)\,d\nu_H(h) = 0, \quad \forall g \in K(G,H(\tau)).$$

Setting $g(x) = p(x)\xi$, with $p \in K(G)$, we get

$$\int_H (\tau(h)\xi,\xi)p(xh)\,d\nu_H(h) = 0.$$

If we make a convenient choice for p, this yields $\xi = 0$, a contradiction. (For instance, let p be non-negative with $\mathrm{supp}(p) \subset xU$, where U is a neighbourhood of the identity in G which satisfies:

$$h \in U \cap H \Rightarrow \mathrm{Re}(\tau(h)\xi,\xi) \geq 0.) \quad \square$$

Next, let μ_1 and μ_2 be two quasi-invariant measures on G/H with continuous R-functions R_1 and R_2. Suppose that τ_1^G and τ_2^G are defined as above, with the use of μ_1 and μ_2 respectively. Denote the corresponding representation spaces by $H_1 = K_\tau^1$ and $H_2 = K_\tau^2$.

LEMMA 2.22. *The representations* τ_1^G *and* τ_2^G *are equivalent.*

PROOF. Let ρ_1 and ρ_2 be the strictly positive continuous functions on G with

$$d\mu_i^{\#}(x) = \rho_i(x)\,d\nu_G(x), \quad i = 1,2.$$

For each f in K_τ^1, define a function Tf on G by

$$(Tf)(x) := \left(\frac{\rho_1(x)}{\rho_2(x)}\right)^{\frac{1}{2}} f(x).$$

Then Tf is compactly supported and continuous , and

$$(Tf)(xh) = \left(\frac{\rho_1(xh)}{\rho_2(xh)}\right)^{\frac{1}{2}} f(xh) = \left(\frac{\rho_1(x)}{\rho_2(x)}\right)^{\frac{1}{2}} f(xh)$$

$$= \tau(h^{-1})(Tf)(x), \qquad \forall h \in H.$$

In the second step we used formula (2.4). Finally,

$$\int_{G/H} \|(Tf)(x)\|^2 d\mu_2(\bar{x}) = \int_{G/H} \|f(x)\|^2 \frac{\rho_1(x)}{\rho_2(x)} d\mu_2(\bar{x})$$
$$= \int_{G/H} \|f(x)\|^2 \, d\mu_1(x).$$

Hence we may consider T as an isometric operator from K_τ^1 into K_τ^2. Clearly, T is surjective and injective. Furthermore,

$$((T\tau_1^G(y)T^{-1})(f))(x) = \left(\frac{\rho_1(x)}{\rho_2(x)}\right)^{\frac{1}{2}} (\tau_1^G(x)T^{-1}f)(x)$$
$$= \left(\frac{\rho_1(x)}{\rho_2(x)}\right)^{\frac{1}{2}} (R_1(\bar{x},y^{-1}))^{\frac{1}{2}}(T^{-1}f)(y^{-1}x)$$
$$= \left(\frac{\rho_1(x)\rho_2(y^{-1}x)}{\rho_2(x)\rho_1(y^{-1}x)}\right)^{\frac{1}{2}} (R_1(\bar{x},y^{-1}))^{\frac{1}{2}}f(y^{-1}x)$$
$$= (R_2(\bar{x},y^{-1}))^{\frac{1}{2}} f(y^{-1}x)$$
$$= (\tau_2^G(y)f)(x), \qquad f \in K_\tau^1.$$

Since K_τ^i lies dense in H_i $(i = 1,2)$, T belongs to $I(\tau_1^G,\tau_2^G)$. □

We will now discuss a slightly different construction of the induced representation space, which sometimes turns out to be more convenient than the one discussed above (e.g. in the next subsection).

Consinder a subspace $\tilde{L}_\tau^2$ of F_τ consisting of the elements f of F_τ which satisfy

(2.10)(i) $x \to (f(x),\xi)$ is measurable for all ξ in $H(\tau)$ (i.e. f is weakly measurable);

(2.10)(ii) $\|f\|^2 := \int_{G/H} \|f(x)\|^2_{H(\tau)} d\mu(\bar{x}) < \infty$.

The second condition requires some explanation. Let $\{\xi_i\}_{i=1}^{\infty}$ be an orthonormal basis for $H(\tau)$. Then

$$\|f(x)\|^2_{H(\tau)} = \sum_{i=1}^{\infty} |(f(x),\xi_i)|^2, \qquad x \in G.$$

Hence, the function $x \longrightarrow \|f(x)\|^2_{H(\tau)}$ is measurable, by virtue of 2.10(i). Moreover, it is constant on left H-cosets by unitarity of τ, so it defines a unique function $\bar{x} \longrightarrow \|f(x)\|^2_{H(\tau)}$ on G/H. Note that for $f,g \in F_\tau$ one has

$$(f(x),g(x)) = \sum_{i=1}^{\infty} (f(x),\xi_i)(\xi_i,g(x)), \qquad x \in G.$$

Hence, the same argument allows us to define a bilinear form in $\tilde{L}_2^\tau$ by

$$(2.11) \qquad (f,g) := \int_{G/H} (f(x),g(x))d\mu(\bar{x}), \qquad f,g \in \tilde{L}^\tau.$$

Identifying functions in $\tilde{L}_2^\tau$ which differ only on a null set, we obtain a new space L_2^τ, in which (2.11) defines a positive definite inner product. It can be shown straightforwardly that L_τ^2 is a Hilbert space and that it is isometrically isomorphic with $\overline{K_\tau}$. Moreover, L_τ^2 is invariant for $\hat{\tau}$ (see (2.9)), and hence $\hat{\tau}$ can be made into a unitary representation of G on L_τ^2, which is equivalent to τ^G. We will denote this representation by τ^G as well.

2.5. A realization of τ^G on $L^2(G/H,H(\tau))$

Let the situation be as in the preceding subsection, and consider again the space L_τ^2. We shall take the elements of this space to be functions on G. Let $L^2(G/H,H(\tau))$ denote the Hilbert space of square μ-integrable functions on G/H with values in $H(\tau)$. Furthermore, choose a Borel cross-section $s\colon G/H \to G$ (see lemma 2.5). For each f in L_τ^2 we define a function g_f on G/H by

$$(2.12)\qquad g_f(\bar{x}) := f(s(\bar{x})), \qquad \bar{x} \in G/H.$$

Obviously, this formula defines an isometric isomorphism from L^2_τ onto $L^2(G/H,H(\tau))$. [Hence, it follows that L^2_τ is a Hilbert space]. This isomorphism can be extended to the corresponding algebras of bounded operators, by setting

$$(\tilde{T}g_f)(\bar{x}) := g_{Tf}(\bar{x}),$$

for any T in $L(L^2_\tau)$. Next we ask ourselves what the induced representation τ^G will look like, when lifted to $L^2(G/H,H(\tau))$. We have

$$\begin{aligned}(\tilde{\tau}^G(y)g_f)(\bar{x}) &= g_{\tau^G(y)f}(\bar{x})\\ &= f(y^{-1}s(\bar{x}))(R(\overline{s(\bar{x})},y^{-1}))^{\frac{1}{2}}.\end{aligned}$$

The element $y^{-1}s(\bar{x})$ belongs to the left H-coset $(y^{-1}x)H$, so there is a unique element h of H such that

$$y^{-1}s(\bar{x}) = s(y^{-1}\bar{x})h.$$

Hence the expression above can be rewritten as

$$f(s(y^{-1}\bar{x})h)(R(\bar{x},y^{-1}))^{\frac{1}{2}} = \tau(h^{-1})g_f(y^{-1}\bar{x})(R(\bar{x},y^{-1}))^{\frac{1}{2}},$$

by (2.7). We define an operator-valued function A: $G/H \times G \longrightarrow U(H(\tau))$ by

$$(2.13)\qquad A(\bar{x},y) := \tau(s(\overline{yx})^{-1}ys(\bar{x})).$$

This function satisfies

$$(2.14)(i)\quad A(\bar{x},e) = I, \qquad \forall\bar{x} \in X;$$

$$(2.14)(ii)\quad A(\bar{x},yz) = A(\overline{zx},y)A(\bar{x},z), \qquad \forall x \in X,\ \forall y,z \in G.$$

(cf. (2.6)(i)). Clearly, A is weakly measurable. The representation of G on $L^2(G/H,\mathcal{H}(\tau))$ which is equivalent to τ^G by the isomorphism (2.12), is thus given by

$$(2.15)\qquad (\tilde{\tau}^G(y)g)(\bar{x}) = A(\overline{y^{-1}x},y)g(y^{-1}\bar{x})(R(\bar{x},y^{-1}))^{\frac{1}{2}}.$$

These considerations lead to an alternative approach to induction. Indeed, let X be a homogeneous space of G and let μ be a quasi-invariant measure on X, with $R\colon X \times G \longrightarrow \mathbb{R}^{>0}$ denoting the corresponding continuous R-function. Furthermore, suppose we are given a weakly measurable operator-valued function

$$A\colon X \times G \longrightarrow \mathcal{U}(\mathcal{H}),$$

where $\mathcal{H}$ is a certain separable Hilbert space. Then we can define operators $T(y)$ on $L^2(X,\mathcal{H},\mu)$ for each y in G by

$$(2.16)\qquad (T(y)f)(\bar{x}) := (R(\bar{x},y^{-1}))^{\frac{1}{2}}A(\overline{y^{-1}x},y)f(y^{-1}\bar{x}).$$

Clearly these operators are well-defined, and, moreover, unitary. Finally, if A satisfies (2.14), then $y \longrightarrow T(y)$ is a homomorphism. In this case, it can be shown that A is of the form (2.13) for a certain Borel cross section $s\colon X \to G$ and a certain unitary representation τ of a closed subgroup H of G with $G/H \approx X$. As a matter of fact, this assertion forms an important stage in the proof of the infinite version of the imprimitivity theorem (§3.1), a sketch of which will be given in §3.2 (cf. also VARADARAJAN [21, thm.9.7]). Another important observation in this context is the following. Let A_1 and A_2 be weakly measurable operator-valued functions from $X \times G$ into $\mathcal{U}(\mathcal{H})$ which satisfy (2.14), and let T_1,T_2 be the representations defined by A_1 and A_2 through (2.16). Then the original representations τ_1 and τ_2 with $\tau_i^G \simeq T_i$, $i = 1,2$, are equivalent if and only if an operator-valued function $C\colon X \to \mathcal{U}(\mathcal{H})$ exists with

$$(2.17)\qquad A_1(\bar{x},y) = C(\overline{yx})^{-1}A_2(\bar{x},y)C(\bar{x}), \qquad \forall x,y \in G.$$

This, again, is part of the proof of the infinite imprimitivity theorem.

REMARK. Note that because of the absence in general of continuous cross-sections from G/H into G, we can not use the space K_τ in this set-up.

Appendix to §2.3.

Proof of lemma 2.15. Let f be a function in $K(G)$ with $f \geq 0$, and set

$$\rho_f(x) := \int_H f(xh) \frac{\Delta_G(h)}{\Delta_H(h)} d\nu_H(h), \qquad x \in G.$$

Then ρ_f defines a positive function on G, which is continuous since f is uniformly continuous. Moreover,

$$\frac{\Delta_G(h_0)}{\Delta_H(h_0)} \rho_f(xh_0) = \int_H f(xh_0h) \frac{\Delta_G(h_0h)}{\Delta_H(h_0h)} d\nu_H(h) = \rho_f(x), \qquad h_0 \in H,$$

so ρ_f satisfies (2.4). However, ρ_f will fail in general to be nonzero. If we admit for f functions which are not in $K(G)$, ρ_f will be well-defined, continuous and strictly positive if

(A.1)(i) f is continuous and $f \geq 0$;

(A.1)(ii) for each x in G there is a compact neighbourhood U of x such that $KH \cap \mathrm{supp}(f)$ is compact;

(A.1)(iii) for all x in G there exists an element h of H such that $f(xh) > 0$.

We will construct such a function by means of an ancillary lemma.

LEMMA. *Let* S *be an open symmetric neighbourhood of the identity in* G*, with compact closure. Then there exists a subset* X(S) *of* G *such that*

(i) *each* H-*coset intersects with at least one set* Sy *for* $y \in X(S)$;

(ii) *for each compact subset* K *of* G*, there are only finitely many* $y \in X(S)$ *such that* $KH \cap Sy \neq \emptyset$.

PROOF. Consider the family of subsets X of G which satisfy the following symmetric condition:

$$\left.\begin{array}{l} x,y \in X \\ x \neq y \end{array}\right\} \Rightarrow x \notin SyH.$$

Note that such a subset always exists. This family is partially ordered by inclusion with each chain having an upper bound. Hence, we can apply the Zorn lemma and choose a maximal set, say X(S). We contend that X(S) meets the qualifications stated in the lemma. First, suppose $xH \cap Sy = \emptyset$ for a certain x in G and all y in X(S). Clearly this contradicts the maximality of X(S). As to (ii), suppose

$$KH \cap Sy_i \neq \emptyset, \qquad y_1,y_2,y_3,\ldots \in X(S),$$

for a certain compact subset K of G. Then there are elements $h_1,h_2,\ldots$ in H with $y_ih_i \in SK$ for all i. Since the closure of SK is compact, this sequence has a cluster point. The set Sy_ih_i is an open neighbourhood of y_ih_i for all i. Thus, for m,n large enough, we must have $y_mh_m \in Sy_nh_n$. But then $y_m \in Sy_nH$, which implies that the sequence $y_1,y_2,\ldots$ contains only finitely many different elements. □

We now finish the proof of lemma 2.15. Let g be a positive function in K(G), with $g(e) \neq 0$, and such that the set $S := \{x \in G;\ g(x) > 0\}$ is symmetric. This is always possible, since we may replace $x \longrightarrow g(x)$ by $x \longrightarrow g(x) + g(x^{-1})$, if need be. Let X(S) be the maximal set corresponding with S, as described in the lemma above, and set $g_y(x) := g(xy^{-1})$, for y in G. Define a function f on G by

$$f(x) := \sum_{y\in X(S)} g_y(x).$$

By virtue of our auxillary lemma, only finitely many functions g_y can be nonzero on a strip KH, for any compact subset K of G. Hence f, being locally the sum of finitely many g_y, is itself continuous. As to (A.1)(ii), we have, for any compact subset K of G:

$$KH \cap \Big(\bigcup_{X(S)} S_y\Big) = \bigcup_{X(S)} (KH\cap S_y) = \bigcup_{i=1}^{n} (KH\cap S_{y_i}),$$

for certain elements $y_1,\ldots,y_n$ in X(S) depending on K. Hence

$$KH \cap \operatorname{supp}(f) = \bigcup_{i=1}^{n} (KH\cap \overline{S}_{y_i}),$$

which is compact. This is even a stronger result then (A.1)(ii). Finally, for any x in G, we have $x \in SyH$ for a certain y in X(S), so $g_y(xh) > 0$ for some h in H. Hence f is nonzero somewhere on each H-coset, which proves (A.1)(iii). □

III. INFINITE IMPRIMITIVITY AND LOCALIZABILITY IN QUANTUM MECHANICS

3.1. The imprimitivity theorem for lcsc. groups

In this subsection we will state the analogue of theorem 1.14 for lcsc. groups. This generalization, which is due to Mackey, will play an essential role in the next chapter, together with theorem 3.4 below.

Let τ be a representation of a finite group G. Recall that a system of imprimitivity (s.o.i.) for τ was defined (in §1.5) to be a family of subspaces $\{V_\gamma\}_{\gamma\in\Gamma}$ of the representation space $V(\tau)$, indexed by a G-space Γ, such that

(i) $V(\tau) = \sum^{\oplus}_{\gamma\in\Gamma} V_\gamma$ as a vector space direct sum;

(ii) $\tau(x)V_\gamma = V_{x(\gamma)}, \quad \forall x \in G, \forall\gamma \in \Gamma.$

We will adjust this definition such as to enable a canonical extension to general topological groups with possibly infinite-dimensional representations.

Consider a family of projections $\{P_\gamma\}_{\gamma\in\Gamma}$ in $V(\tau)$ such that

$$P_\gamma(V(\tau)) = V_\gamma, \qquad \forall\gamma \in \Gamma.$$

Note that Γ as a finite set has a trivial Borel structure, generated by its discrete topology. In other words, the Borel sets of Γ are just its subsets.

If we define a projection P_E in $V(\tau)$ for each subset E of Γ by

$$P_E := \sum_{\gamma\in E} P_\gamma,$$

(in particular $P_{\{\gamma\}} = P_\gamma$) then it is clear that the mapping $P\colon E \to P_E$ satisfies

$$\begin{cases} P_\Gamma = I \\ P_{E\cap F} = P_E \cdot P_F, & \forall E,F \subset \Gamma \\ \Sigma P_{E_i} = P_{\cup E_i}, & \forall E_1,E_2,\ldots \subset \Gamma \text{ with } E_i \cap E_j = \emptyset \text{ for } i \neq j. \end{cases}$$

Hence, P is a so-called projection-valued (or spectral) measure on Γ, acting in $V(\tau)$. Let $\xi = \sum_{\gamma\in\Gamma} \xi_\gamma$ be the decomposition of an element ξ of $V(\tau)$ into V_γ-components. Then, by virtue of property (ii) above, we have

$$\tau(x)P_{\gamma_0}\tau(x^{-1})\xi = \tau(x)P_{\gamma_0} \sum_{\gamma\in\Gamma} (\tau(x^{-1})\xi)_{x^{-1}(\gamma)}$$

$$= \tau(x)(\tau(x^{-1})\xi)_{\gamma_0} = \xi_{x(\gamma_0)} = P_{x(\gamma_0)}\xi.$$

By linearity, this implies

$$\text{(3.1)} \qquad \tau(x)P_E\,\tau(x^{-1}) = P_{x[E]} ,$$

for all x in G and all (Borel) subsets E of Γ, where $x[E] := \{x(\gamma);\gamma \in E\}$.

Conversely, if we are given a finite group G with a representation τ, a G-space Γ and a projection-valued measure P, based on Γ and acting in $V(\tau)$, such that τ and P are related by (3.1), then it is clear that the collection $\{P_{\{\gamma\}}(V(\tau))\}_{\gamma\in\Gamma}$ forms a s.o.i. for τ.

These considerations lead us to the following definition.

DEFINITION 3.1. Let G be a topological group. A *system of imprimitivity* (s.o.i.) for G acting in a Hilbert space H, is a triple (Γ,τ,P), where

(i) Γ is a continuous G-space;

(ii) τ is a unitary representation of G on H;

(iii) P is a projection-valued measure on Γ, acting in H, such that for all Borel subsets E of Γ and all x in G:

$$\tau(x)P_E\tau(x)^{-1} = P_{x[E]}.$$

As in the finite case, the system is said to be *transitive (trivial)* according to Γ being a *transitive (trivial)* G-space.

Many properties of representations may be formulated in terms of imprimitivity systems as well. Instead of the intertwining space $I(\tau,\sigma)$ of two representations τ and σ of G, we can consider the intertwining space of two s.o.i. based on G-homeomorphic G-spaces, say (Γ,τ,P) and (Δ,σ,Q), denoted by $I((\tau,P),(\sigma,Q))$. This space is defined to consist of all operators T: $\mathcal{H}(\tau) \to \mathcal{H}(\sigma)$, which satisfy

$$(3.2)(i) \qquad T\tau(x) = \sigma(x)T, \quad \forall x \in G;$$

$$(3.2)(ii) \qquad TP_E = Q_{\Phi(E)}T, \quad \text{for all Borel sets } E \text{ in } \Gamma.$$

Here Φ: $\Gamma \to \Delta$ denotes the G-homeomorphism of Γ onto Δ. Thus, if we denote by $I(P,Q)$ the space of operators T which satisfy (3.2)(ii), we have

$$I((\tau,P),(\sigma,Q)) := I(\tau,\sigma) \cap I(P,Q).$$

Two systems (Γ,τ,P) and (Δ,σ,Q) for G are said to be *equivalent* if

(i) Γ and Δ are G-homeomorphic;

(ii) $I((\tau,P),(\sigma,Q))$ contains an isometrical isomorphism.

Finally we shall say that a s.o.i. (Γ,τ,P) is *irreducible* if the sets of operators $\{\tau(x);\ x \in G\}$ and $\{P_E;\ E \text{ a Borel set in } \Gamma\}$ have no common nontrivial invariant subspaces. This is equivalent to the condition:

$$(3.3) \qquad I(\tau,P) \ (:= I((\tau,P),(\tau,P))) = \{\lambda I;\ \lambda \in \mathbb{C}\}.$$

<u>EXAMPLE 3.2</u>. Let λ be the regular representation of a lcsc. group G, on $L^2(G)$. Define a projection P_E for each Borel set E in G by

$$(3.4) \qquad (P_E(f))(x) := \chi_E(x)f(x), \qquad f \in L^2(G),$$

where χ_E denotes as usual the characteristic function of E. Obviously, relation (3.1) holds with τ replaced by λ. Therefore (G,λ,P) is a (transitive) s.o.i. for G, where G is considered as a continuous G-space by left translation (cf. example 1.13).

EXAMPLE 3.3. Suppose that τ is a unitary representation of a locally compact second countable group G, which is induced on G by a unitary representation σ of a certain closed subgroup H of G. Thus, the space $\mathcal{H}(\sigma)$ consists of $\mathcal{H}(\sigma)$-valued functions on G (cf. §2.4). We define a projection-valued measure P, based on the coset space G/H, and acting in $\mathcal{H}(\tau)$, by

$$(3.5)\qquad (P_E(f))(x) := \chi_E(\bar{x})f(x), \qquad E \text{ Borel set in } G/H.$$

Using the definition of an induced representation (formula (2.9)) one finds:

$$\begin{aligned}(\tau(x)P_E\tau(x)^{-1}f)(y) &= (P_E\tau(x)^{-1}f)(x^{-1}y)(R(\bar{y},x^{-1}))^{\frac{1}{2}} \\ &= \chi_E(\overline{x^{-1}y})(\tau(x)^{-1}f)(x^{-1}y)(R(\bar{y},x^{-1}))^{\frac{1}{2}} \\ &= \chi_{x[E]}(\bar{y})f(y)(R(x^{-1}\bar{y},x)R(\bar{y},x^{-1}))^{\frac{1}{2}} \\ &= \chi_{x[E]}(\bar{y})f(y) = (P_{x[E]}f)(y).\end{aligned}$$

Consequently, $(G/H,\tau,P)$ is a (transitive) s.o.i. admitted by τ. We shall call this system *canonically associated* with the induced representation $\sigma^G = \tau$, or, shortly, the canonical system of σ^G.

There exists an interesting relationship between an induced representation and its canonical system. Using the notation of example 3.3, let T be a bounded operator on $\mathcal{H}(\sigma)$, which is a member of the commuting algebra $I(\sigma)$. By means of T we can define an operator $\hat{T}$ on $\mathcal{H}(\sigma^G)$, by

$$(3.6)\qquad (\hat{T}f)(x) := Tf(x), \qquad f \in \mathcal{H}(\sigma^G).$$

If x and y are elements of G and H respectively, we have

$$(\hat{T}f)(xy^{-1}) = T\sigma(y)f(x) = \sigma(y)Tf(x) = \sigma(y)(\hat{T}f)(x),$$

since T belongs to $I(\sigma)$. Furthermore, from the definition of the inner product in $\mathcal{H}(\sigma^G)$, it follows at once that

$$\|\hat{T}f\| \leq \|T\| \cdot \|f\|, \qquad f \in H(\sigma^G).$$

(Obviously, $\hat{T}f$ is weakly measurable). Hence, $\hat{T}$ is a bounded operator on $H(\sigma^G)$. Concerning the mapping $T \to \hat{T}$ we can state the following important theorem:

THEOREM 3.4. *The mapping* $T \to \hat{T}$, *defined by formula* (3.6) *maps the commuting algebra* $I(\sigma)$ *isomorphically onto the commuting algebra* $I(\sigma^G, P)$ *of the canonical system of* σ^G.

PROOF. We show successively that the image of $T \to \hat{T}$ lies in $I(\sigma^G, P)$, that the mapping is injective and that it is surjective. (Linearity is clear).

Let T belong to $I(\sigma)$. Then, for each Borel set E in G, we have

$$(\hat{T}(\chi_E \cdot f))(x) = T(\chi_E(x) \cdot f(x)) = \chi_E(x) Tf(x) = \chi_E(x)(\hat{T}f)(x),$$

$$f \in H(\sigma^G).$$

Since $\chi_F(\bar{x}) = \chi_{F \cdot H}(x)$, if F is a Borel set in G/H, the operator $\hat{T}$ commutes with the projection-valued measure occurring in the canonical system of σ^G, which is given by formula (3.5). Moreover, we have

$$\begin{aligned}(\hat{T}\sigma^G(y)f)(x) &= T(f(y^{-1}x)(R(\bar{x},y^{-1}))^{\frac{1}{2}}) \\ &= (R(\bar{x},y^{-1}))^{\frac{1}{2}} Tf(y^{-1}x) = (R(\bar{x},y^{-1}))^{\frac{1}{2}}(\hat{T}f)(y^{-1}x) \\ &= (\sigma^G(y)\hat{T}f)(x), \qquad f \in H(\sigma^G),\end{aligned}$$

since R is a real-valued function. Therefore, $\hat{T}$ belongs to $I(\sigma^G)$ as well, which proves our first assertion.

As to injectivity, this follows at once from corollary 2.21, which stated that for any x in G, the subset $\{f(x);\ f \in K_\sigma\}$ lies dense in $H(\sigma)$. Indeed, if $Tf(x) = Sf(x)$ for two bounded operators T and S on $H(\sigma)$, and all f in K_σ, then $T = S$. Since K_σ can be considered as a (dense) subspace of $H(\sigma^G)$, this proves injectivity of $T \to \hat{T}$.

The proof of surjectivity onto $I(\sigma^G, P)$ is somewhat more complicated, and for the sake of continuity it will be given in an appendix to this chapter. ($\square$)

<u>COROLLARY 3.5</u>. *The canonical system of an induced representation* σ^G *of a lcsc. group is irreducible if and only if the original representation* σ *is irreducible.*

<u>PROOF</u>. Clear. $\square$

Next, we state the general imprimitivity theorem.

<u>THEOREM 3.6</u>. (MACKEY). *Let* τ *be a unitary representation of a lcsc. group* G, *and let* H *be an arbitrary closed subgroup of* G. *Then the existence of a transitive system of imprimitivity* $(G/H,\tau,P)$ *implies the existence of a unitary representation* σ *of* H *such that* $(G/H,\tau,P)$ *is equivalent to the system canonically associated with* σ^G. *In particular,* τ *is equivalent to* σ^G. *Moreover, the equivalence class of* σ *is completely determined by the system* $(G/H,\tau,P)$.

The proof of this theorem is rather complicated. There exist several variants, of which the most recent ones (BARUT & RĄCZKA [2], KIRILLOV [9]) are based purely on functional analysis. The original proofs of MACKEY (see [11], [13], or [14] for a sketch) are maybe not very accessible, in that they leave a lot to the imagination. However, they are based on some essential ideas, which play a fundamental (though not very perceptible) role in the work of Mackay on induction for locally compact groups. The ideas are connected with the "classical" cohomology theory of groups (EILENBERG/MACLANE).

In the next subsection we will try and sketch these ideas, following VARADARAJAN [21], and show how the imprimitivity theorem can be derived from them.

The theorem has proved to be amenable to generalizations in many directions. One has to start with extending the concept of induction to larger classes of groups on the one hand, and larger classes of representations on the other hand. For instance, second countability of G and separability of the representation space (which we use as a convention) can be omitted from the theorem. Furthermore, after a suitable reformulation, the theorem keeps its validity for so-called projection or multiplier representations (see MACKEY [13]).

3.2 On a proof of the imprimitivity theorem

Let G be a lcsc. group, and let $H \subset G$ be a closed subgroup, throughout this subsection. We set $X := G/H$. M will denote a second countable Hausdorff group, until further specifications.

DEFINITION 3.7. A Borel map $f\colon X \times G \to M$ is called a (X,G,M)-*cocycle* if it satisfies

$$(3.7)(i) \qquad f(\bar{x},e) = I, \qquad \forall x \in G;$$

$$(3.6)(ii) \qquad f(\bar{x},yz) = f(\overline{zx},y)f(\bar{x},z), \qquad \forall x,y,z \in G.$$

(Here I denotes the identity in M).

DEFINITION 3.8. Two (X,G,M)-cocycles f_1 and f_2 are said to be *cohomologous* ($f_1 \simeq f_2$) if there exists a Borel map $b\colon X \to M$ such that

$$(3.8) \qquad f_1(\bar{x},y) = b(\overline{yx})^{-1} f_2(\bar{x},y) b(\bar{x}), \qquad \forall x,y \in G.$$

Obviously, (3.8) defines an equivalence relation in the set of all (X,G,M)-cocycles. Equivalence classes are called (X,G,M)-*cohomology classes*.

REMARK. In VARADARAJAN [21, p.27], the functions of definition 3.7 are called strict cocycles, and the relation between f_1 and f_2 given by (3.8) is called strict cohomologous. His definitions of cocycles and cohomologous admit deviations on null-sets in the identities (3.7)(i), (ii) and (3.8) (that is, null-sets in X, $X \times G \times G$ and $X \times G$, respectively, w.r.t. Haar measure on G and quasi-invariant measure on X). In view of lemma 8.26 and part of theorem 8.27 in [21], which state respectively:

- each cocycle (in the sense of [21]) is a.e. (on $X \times G$) equal to a strict cocycle, which is unique up to strict cohomology;
- each cohomology class (sic) contains a unique, nonvoid, strict cohomology class;

we feel justified to circumvent measure theoretical details and use definitions 3.7 and 3.8. (However, in a certain part of the proof of the imprimitivity theorem, the use of cocycles in the sense of [21] can not be avoided. We will omit this part.)

Let f be a (X,G,M)-cocycle. Then the map τ: $H \to M$, defined by $\tau(h) := f(\bar{e},h)$, is clearly a Borel homomorphism. We will call τ the *homomorphism associated with* f. Two homomorphisms τ_i: $H \to M$ $(i = 1,2)$ are called equivalent $(\tau_1 \simeq \tau_2)$ if there exists an element T of M with

$$T\tau_1(h) = \tau_2(h)T, \qquad \forall h \in H.$$

By virtue of lemma 8.28 in [21] (mentioned before in our §2.1), any Borel homomorphism from a lcsc. group into a second countable Hausdorff group is automatically continuous. The reader should keep this in mind, since we are going to use this fact later on, when we take M to be the unitary group of a separable Hilbert space and call τ a representation of H.

The following theorem relates (X,G,M)-cohomology classes to equivalence classes of Borel homomorphisms from H into M, and constitutes the first important step towards the proof of the imprimitivity theorem. If γ is a (X,G,M)-cohomology class, we let $\tilde{\gamma}$ denote the set of continuous homomorphisms associated with the elements of γ.

THEOREM 3.9. *The assignment* $\gamma \to \tilde{\gamma}$ *establishes a one-to-one correspondence between the set of all* (X,G,M)*-cohomology classes and the set of all equivalence classes of continuous homomorphisms from* H *into* M.

PROOF. We can split the proof into two parts:

(i) Let f_1 and f_2 be (X,G,M)-cocycles and let τ_1 and τ_2 be the associated homomorphisms. Then $f_1 \simeq f_2$ iff $\tau_1 \simeq \tau_2$.

(ii) Each continuous homomorphism τ: $H \to M$ is associated with a certain (X,G,M)-cocycle.

Let f_1 and f_2 be two (X,G,M)-cocycles and let τ_1 and τ_2 be the associated homomorphisms. We define two Borel maps $b_i\colon G \to M$ by

$$b_i(y) := f_i(\bar{e},y), \qquad y \in G.$$

From this definition, we have

$$(3.9) \qquad b_i(xh) = b_i(x)\tau_i(h), \quad \forall x \in G,\ h \in H,$$

and

$$(3.10) \qquad f_i(\bar{x},y) = b_i(yx)b_i(x)^{-1}, \qquad \forall x,y \in G,$$

as can be easily checked.

Now, suppose that for some $T \in M$, we have

$$T\tau_1(h) = \tau_2(h)T, \qquad \forall h \in H.$$

Then, from (3.9) it follows that

$$b_2(x)Tb_1(x)^{-1} = b_2(xh)Tb_1(xh)^{-1}, \quad \forall x \in G,\ h \in H.$$

Hence, a unique Borel map $b\colon X \to M$ exists, with

$$b(\bar{x}) = b_2(x)Tb_1(x)^{-1}, \quad x \in G.$$

Using the properties of cocycles and identity (3.10), an easy calculation yields

$$(3.11) \qquad f_2(\bar{x},y) = b(\overline{yx})f_1(\bar{x},y)b(\bar{x})^{-1}, \quad \forall x,y \in G.$$

Thus, $f_1 \simeq f_2$.

Conversely, suppose $f_1 \simeq f_2$, and let this equivalence be established by a Borel map $b\colon X \to M$. Then (3.8) yields

$$f_1(\bar{e},h) = b(\bar{e})^{-1}f_2(\bar{e},h)b(\bar{e})^{-1}, \quad \forall h \in H.$$

Thus, setting T := b($\bar{e}$), we obtain

$$T\tau_1(h) = \tau_2(e)T, \qquad \forall h \in H.$$

This finishes the first part of the proof. As to surjectivity of $\gamma \to \tilde{\gamma}$, let τ be a continuous homomorphism from H into M. We choose a Borel cross-section s: X $\to$ G, with s($\bar{e}$) = e (this is legal, since for any Borel cross-section s we may define a new Borel cross-section s' by setting $s'(\bar{x}) = s(\bar{x})\cdot s(\bar{e})^{-1}$). Next, we define a Borel map f: X $\times$ G $\to$ M, by

$$f(\bar{x},y) := \tau(s(\overline{yx})^{-1}ys(\bar{x})).$$

A brief calculation shows that f is a well-defined (X,G,M)-cocycle, and, moreover, since s($\bar{e}$) = e, we have

$$f(\bar{e},h) = \tau(h). \qquad \square$$

The next theorem implies the imprimitivity theorem, and explains at the same time the idea behind the discussion in this subsection. First we introduce a few notations, which will be sustained till the end of this subsection. H_n will denote a fixed Hilbert space of dimension n = ∞,1,2,..., and M_n will denote its unitary group, provided with the weak topology. Furthermore, we fix a quasi-invariant measure μ on X, and set $K_n := L^2(X,H_n,\mu)$. In K_n we define a projection-valued measure P^n, based on X, by

$$P^n_E f = \chi_E f, \qquad E \in \mathcal{B}(X).$$

Note that the equivalence class of P^n is independent of our choice of a quasi-invariant measure on X. The proof of this assertion is similar to the one of lemma 2.22. [Recall that two projection-valued measures P and Q acting in Hilbert spaces H and H', respectively, and based on a Borel space B, are said to be equivalent if there exists an isometric isomorphism T: $H \to H'$ such that $TP_E = Q_E T$, $\forall E \in \mathcal{B}(B)$.]

THEOREM 3.10.

(i) *Any system of imprimitivity* (X,τ,P) *for* G *is equivalent to a system of the form* (X,τ',P^n)*, for a unique* $n \in \{\infty,1,2,\ldots\}$.

(ii) *There exists a one-to-one map* $\gamma \to \Sigma(\gamma)$ *from the set of* (X,G,M_n)*-cocycles onto the set of equivalence classes of systems of imprimitivity of the form* (X,τ,P^n).

(iii) *The map* $\gamma \to \Sigma(\gamma)$ *enjoys the following property: Let* τ *be the homomorphism determined by an element of* γ*. Then any system in* $\Sigma(\gamma)$ *is equivalent to the canonical system associated with the induced representation* τ^G.

Before giving the proof of this theorem, we will state an important result from spectral multiplicity theory, and deduce a lemma from it which applies to our situation. This is done in order to obtain part (i) of the above theorem. The first mentioned result can be found in e.g. HALMOS [6, chapter III, particularly §67 and 68.].

Let Y be a second countable Hausdorff space. For any finite Borel measure ν on Y we set $K_{n,\nu} := L^2(Y,H_n,\nu)$. Furthermore, let $P^{n,\nu}$ denote the projection-valued measure based on Y and acting in $K_{n,\nu}$ by

$$P_E^{n,\nu} f = \chi_E f, \qquad E \in \mathcal{B}(Y).$$

It will turn out that the measures $P^{n,\nu}$ are the canonical building blocks for arbitrary projection-valued measures on Y. Indeed, let $\nu_\infty,\nu_1,\nu_2,\ldots$ be a sequence of mutually singular finite Borel measures on Y (recall that two measures μ and ν on Y are called mutually singular, notation $\mu \perp \nu$, if $\mu(B) = \nu(Y-B) = 0$ for some Borel set B in Y.). We set

$$K = \bigoplus_n K_{n,\nu_n},$$

and define a projection-valued measure $P = P(\{H_n\},\{\nu_n\})$ acting in K and based on Y, by

$$P_E[(f_\infty,f_1,f_2,\ldots)] := (P^{\infty,\nu_\infty} f_\infty, P_E^{1,\nu_1} f_1,\ldots), \qquad E \in \mathcal{B}(Y).$$

THEOREM 3.11. *Any projection-valued measure* P *on* Y *determines a unique sequence* $[\mu_\infty]$, $[\mu_1]$, $[\mu_2]$,... *of mutually singular measure classes on* Y *such that*

$$\nu_n \in [\mu_n],\ n = \infty,1,2,\ldots \Leftrightarrow P \simeq P(\{H_n\},\{\nu_n\}).$$

In particular $P(\{H_n\},\{\nu_n\}) \simeq P(\{H_n\},\{\nu_n'\})$ *implies* $\nu_n \simeq \nu_n'$, $n = \infty,1,2,\ldots$

Now, we consider the consequences of this theorem in the present situation, that is, with Y = X, and P being part of a system of imprimitivity (x,τ,P) for G. We call a projection-valued measure *homogeneous* if all but one of the measure classes it determines are zero.

LEMMA 3.12. *Let* (X,τ,P) *be a system of imprimitivity for* G *acting in a Hilbert space* H. *Then* P *is homogeneous. Moreover, the only nonzero measure class associated with* P *is the class of quasi-invariant measures on* X.

PROOF. Suppose $P \simeq P(\{H_n\},\{\nu_n\})$ for some sequence $\nu_\infty,\nu_1,\nu_2,\ldots$ of mutually singular Borel measures on X. Let x be any element of G, and set $Q_E := P_{x[E]}$, $E \in B(X)$. Then $Q: E \to Q_E$ is a projection-valued measure on X, which is equivalent to P, since

$$\tau(x)P_E\tau(x)^{-1} = P_{x[E]} = Q_E, \qquad \forall E \in B(X).$$

On the other hand, we have

$$Q \simeq P(\{H_n\},\{(\nu_n)_x\}),$$

where $(\nu_n)_x$ denotes the translated measure $(\nu_n)_x(E) := \nu_n(x[E])$, $E \in B(X)$. (Note that $\nu_n \perp \nu_m$, $n \neq m$, implies $(\nu_n)_x \perp (\nu_m)_x$). This can be seen as follows. Define a linear map

$$U := \oplus_n K_{n,\nu_n} \longrightarrow \oplus_n K_{n,(\nu_n)_x},$$

by

$$U[(f_\infty,f_1,\ldots)] := ((f_\infty)_x,(f_1)_x,\ldots),$$

where $(f_n)_x(y) := f_n(xy)$. Obviously, U establishes an isometric isomorphism. Write $P^* := P(\{H_n\},\{\nu_n\})$ and $P^{*,x} := P(\{H_n\},\{(\nu_n)_x\})$. There exists an isometric isomorphism

$$T: H \longrightarrow \bigoplus_n K_{n,\nu_n}$$

with

$$TP_E = P_E^* T, \qquad \forall E \in \mathcal{B}(X).$$

We have

$$UTQ_E = P_E^{*,x} UT, \quad \forall E \in \mathcal{B}(X),$$

as can be readily verified. By virtue of the preceding theorem, $\nu_n \simeq (\nu_n)_x$, $n = \infty,1,2,\ldots$, which implies, x being arbitrary, that all measure classes determined by P are invariant. But then P must be homogeneous, since quasi-invariant measures can not be mutually singular. □

We now proceed to the proof of Theorem 3.10. Let (X,τ,P) be a system of imprimitivity for G acting in H. Then $P \simeq P^n$, for a certain $n \in \{\infty,1,2,\ldots\}$, on account of the preceding lemma. Thus, there exists an isometric isomorphism $T: H \to K_n$ such that

$$TP_E = P_E^n T, \qquad \forall E \in \mathcal{B}(X).$$

Define a new representation τ' of G, on K_n, by

$$\tau'(x) := T\tau(x)T^{-1}, \qquad x \in G.$$

Then (X,τ',P^n) is a system of imprimitivity, and equivalent to (X,τ,P). This proves part (i) of Theorem 3.10.

Next, let ϕ be a (X,G,M)-cocycle and define a representation τ of G on K_n by

$$(\tau(x)f)(\bar{y}) = (R(\bar{y},x^{-1}))^{\frac{1}{2}}\phi(\overline{x^{-1}y},x)f(\overline{x^{-1}y}),$$

where R is a continuous R-function corresponding to μ. It can be easily verified that τ is a well-defined unitary representation of G. Furthermore

$$(\tau(x)P_E^n\tau(x^{-1})f)(y)$$

$$= (R(\bar{y},x^{-1}))^{\frac{1}{2}}\phi(x^{-1}y,x)(P_E^n\tau(x^{-1})f)(x^{-1}y)$$

$$= \chi_E(x^{-1}y)(R(\bar{y},x^{-1}))^{\frac{1}{2}}\phi(x^{-1}y,x)(\tau(x^{-1})f)(x^{-1}y)$$

$$= \chi_E(x^{-1}y)(R(\bar{y},x^{-1})R(x^{-1}y,x))^{\frac{1}{2}}\phi(x^{-1}y,x)\phi(\bar{y},x^{-1})f(\bar{y})$$

$$= \chi_{x[E]}(\bar{y})\phi(\bar{y},e)f(\bar{y})$$

$$= \chi_{x[E]}(\bar{y})f(\bar{y}) = (P_{x[E]}^n f)(\bar{y}).$$

Hence, (X,τ,P^n) is a system of imprimitivity for G. The equivalence class of this system does not depend on our choice of μ, as can be proved along the same lines as lemma 2.22. It is also not affected if we choose another co-cycle in the cohomology class of ϕ. Indeed suppose $\phi' \simeq \phi$, and let $b\colon X \to M_n$ be a Borel map with

$$\phi(\bar{x},y) = b(\overline{yx})^{-1}\phi'(\bar{x},y)b(\bar{x}), \qquad \forall x,y \in G.$$

If we set

$$(Bf)(\bar{x}) := b(\bar{x})f(\bar{x}), \qquad f \in \mathcal{K}_n,$$

then it is clear that B defines a unitary operator on $\mathcal{K}_n$. Moreover, we have

(3.12)(i) $\quad BP_E^n = P_E^n B, \qquad \forall E \in \mathcal{B}(X);$

(3.12)(ii) $\quad B\tau(x) = \tau'(x)B,$

where τ' is the representation on $\mathcal{K}_n$ defined by ϕ'. (3.12)(i) is trivial, and (3.12)(ii) follows from the following easy computation:

$$(B\tau(x)B^{-1}f)(\bar{y}) =$$

$$(R(\bar{y},x^{-1}))^{\frac{1}{2}}b(\bar{y})\phi(\overline{x^{-1}y},x)b(\overline{x^{-1}y})f(\overline{x^{-1}y})$$

$$= (R(\bar{y},x^{-1}))^{\frac{1}{2}}\phi'(\overline{x^{-1}y},x)f(\overline{x^{-1}y})$$

$$= (\tau'(x)f)(\bar{y}).$$

Consequently, $(X,\tau,P^n) \simeq (X,\tau',P^n)$. Thus, we have constructed a map $\gamma \to \Sigma(\gamma)$ from the set of (X,G,M_n)-cohomology classes into the set of equivalence classes of systems of imprimitivity of the form (X,τ,P^n). Proving surjectivity of this map requires some technicalities which have no direct relationship to our subject matter, and are therefore omitted. We refer the reader to [21, thm. 9.11].

As to part (iii) of theorem 3.10, let γ be a (G,X,M_n)-cohomology class, and let ϕ belong to γ. The representation τ of H on H_n associated with ϕ is given by $\tau(h) = \phi(\bar{e},h)$, $h \in H$. Let $s\colon X \to G$ be a Borel cross-section with $s(\bar{e}) = e$, and set

$$\phi'(\bar{x},y) = \tau(s(\overline{yx})^{-1}ys(\bar{x})), \qquad x,y \in G.$$

Then ϕ' is a (G,X,M_n)-cocycle which is cohomologous to ϕ (theorem 3.9). In §2.5 we have seen that the formula

$$(\tilde{\tau}^G(x)f)(\bar{y}) = (R(\bar{y},x^{-1}))^{\frac{1}{2}}\phi'(\overline{x^{-1}y},x)f(\overline{x^{-1}y})$$

defines a realization $\tilde{\tau}^G$ of the induced representation τ^G on $L^2(X,H_n,\mu)$. Let (X,τ^G,P) be the canonical system associated with τ^G. Then the isometric isomorphism $f \to g_f$ from $H(\tau^G)$ onto $L^2(X,H_n,\mu)$ given by

$$g_f(\bar{x}) := f(s(\bar{x})),$$

is easily seen to establish equivalence between (X,τ^G,P) and $(X,\tilde{\tau}^G,P^n)$. Indeed, we have

$$g_{\tau^G(x)f} = \tilde{\tau}^G g_f, \qquad x \in G,$$

by definition of $\tilde{\tau}^G$, and

$$\begin{aligned} g_{P_E f}(\bar{x}) &= \chi_E(\overline{s(\bar{x})}) f(s(\bar{x})) \\ &= \chi_E(\bar{x}) f(s(\bar{x})) \\ &= P_E^n g_f(\bar{x}), \qquad x \in G,\ E \in \mathcal{B}(X), \end{aligned}$$

by definition of the projection-valued measure in the canonical system. On the other hand, since $\phi \simeq \phi'$, we have also $(X, \tilde{\tau}^G, P^n) \simeq (X, \sigma, P^n)$, where (X, σ, P^n) denotes the system defined by ϕ. This finishes the proof of theorem 3.10.

3.3 Localizability in quantum mechanics

It was discovered independently by Mackey and the physicist Wightman that the notion of imprimitivity can be employed in giving a mathematically rigorous description of the difficult physical concept of localizability. The physical background of this result can be traced back to a paper written by NEWTON & WIGNER [24], in which the localizability concept was approached from a rather heuristic point of view. By coincidence this paper was published in the same year (1949) as was the first paper of MACKEY's [11] on imprimitivity, which provided the tools to repair the mathematical shortcomings of [24]. We will try to sketch the relationship between imprimitivity and localizability, and in doing so we will more or less follow the exposition of WIGHTMAN [22].

It has to be understood that the concept of localizability as we will develop it, is of a rather academical nature, and can serve only as a basis for a physically consistent theory of "measurement of position".

Consider a relativistic system in the Minkowski space-time M. We denote this system by S. There is associated with S a unitary representation of the continuous Poincaré group $P_+^\uparrow$ on the space of states of S. This representation is possibly a projective representation with phase-factor -1.

This can be remedied by considering the covering group $\tilde{P}_+^\uparrow$ of $P_+^\uparrow$, but we will assume that we are dealing with a proper unitary representation of $P_+^\uparrow$, and at the end of our discussion we will make a remark on the projective case. Denote this representation by U: $x \to U(x)$, and let $\mathcal{H} = \mathcal{H}(U)$ be the space of states of S.

We assume that the system S is localizable somewhere in the space $\mathbb{R}^3 \subset M$ at a fixed time. That is, there exist well-defined observables corresponding with the measurement of the position of S in any state in the various parts of $\mathbb{R}^3$. If B is a Borel subset of $\mathbb{R}^3$, we denote the self-adjoint operator corresponding to the observable measuring the position of S in B by E(B). Then $E(B)\psi = \psi$ if S, being in the state ψ, is localized inside B, and $E(S)\psi = 0$ if it is not. Clearly, the only eigenvalues of E(B) are zero and one. Together with its self-adjointness this implies that E(B) is a projection in $\mathcal{H}$. We now give a set of axioms for the family

$$\{E(B);\ B \in \mathcal{B}(\mathbb{R}^3)\},$$

where $\mathcal{B}(\mathbb{R}^3)$ denotes the σ-algebra of Borel subsets of $\mathbb{R}^3$. These axioms express just the reasonable expectations one would have from a well-grounded notion of localization. That we choose the family of Borel sets as our point of departure is not as strange as it maybe seems to be; this is explained in appendix I of WIGHTMAN [22].

Axioms

I *For each* B *in* $\mathcal{B}(\mathbb{R}^3)$, *the operator* E(B) *on* $\mathcal{H}$ *is well-defined.*

II *If* B_1 *and* B_2 *are disjoint Borel sets, then the system can be localized only in one of* B_1 *and* B_2, *i.e.*

$$B_1 \cap B_2 = \emptyset \Rightarrow E(B_1)E(B_2) = 0.$$

III *The set of states in which the system is localized in a union* $\cup_i B_i$ *of Borel sets is the linear span of the states in which the system is localized in one of the sets* B_i. *Together with* II *this means*

$$B_i \cap B_j = \emptyset \Rightarrow E(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} E(B_i).$$

IV *In each state the system can be localized in* $\mathbb{R}^3$, *i.e.*

$$E(\mathbb{R}^3) = I.$$

V *The measurement of position is in a sense invariant (or, rather, covariant) under Euclidean transformations of* $\mathbb{R}^3$. We explain this below.

Notice that II and III imply

$$E(B_1)E(B_2) = E(B_1 \cap B_2) = E(B_2)E(B_1), \quad \forall B_1, B_2 \in \mathcal{B}(\mathbb{R}^3).$$

In this set-up the number $p(B)$, defined by

$$p(B) = \frac{\|E(B)\psi\|^2}{\|\psi\|^2} ,$$

is equal to the probability of finding S inside B, if it is in the state ψ.

The first four axioms imply that the mapping $E: B \to E(B)$ from $\mathcal{B}(\mathbb{R}^3)$ into $L(H)$ is a projection valued measure on $\mathbb{R}^3$. The fifth one provides us with a relationship between this measure and the representation U of $P_+^\uparrow$, associated with S. Indeed, let E(3) denote the group of Euclidean motions of $\mathbb{R}^3$, then E(3) is a subgroup of $P_+^\uparrow$, which is to be interpreted as the pointwise stabilizing subgroup of the time-axis in M. By $V: x \to V(x)$ we denote the unitary representation of E(3) obtained by restricting U to E(3). Thus, the operators $V(x)$ give the symmetries of H corresponding to the Euclidean transformation of $\mathbb{R}^3$. Now axiom V expresses that if ψ is a state in which S is localized inside a Borel set B, then the state $V(x)\psi$ in which S is after transformation of the space by x, is a state for which S is localized in the transformed set $x[B]$. In formula, this means

$$E(B)\psi = \psi \iff E(x[B])V(x)\psi = V(x)\psi.$$

Since this equivalence is valid for all states, we infer the identity

$$V(x)E(B)V(x)^{-1} = E(x[B]), \quad \forall x \in E(3),\ \forall B \in \mathcal{B}(\mathbb{R}^3).$$

But this expression implies exactly that the triple $(\mathbb{R}^3, V, E)$ is a system of imprimitivity for E(3). Moreover, this system is transitive, since $\mathbb{R}^3$ is a homogeneous space of E(3). Notice that the stabilizer in E(3) at any point of $\mathbb{R}^3$ is isomorphic with SO(3).

Application of the imprimitivity theorem yields the following results:

(i) V is induced on E(3) by a certain representation τ of SO(3);

(ii) The space H can be identified with a space $L^2(\mathbb{R}^3, H(\tau))$, such that V acts in this space by

$$(V((y,R))f(x) = \tau(R)f(R^{-1}(x-y)), \quad x,y \in \mathbb{R}^3, \quad R \in SO(3).$$

(iii) E is equivalent to the projection-valued measure in $L^2(\mathbb{R}^3, H(\tau))$ defined by multiplication with characteristic functions.

From these facts, the first one is perhaps the most striking. Indeed, the restriction of U to E(3) being induced from SO(3) is a stringent condition on U. Since the only assumption we have made, is localizability of the system S, we can derive from (i) a criterion of localizability:

<u>CRITERION</u>. *A relativistic system in the Minkowski space-time is localizable in* $\mathbb{R}^3$ *if and only if the corresponding unitary representation of the continuous Poincaré group on the space of states, restricted to the Euclidean group* E(3), *is induced on* E(3) *from its subgroup* SO(3).

The problem now arises of determining which representations of $P_+^\uparrow$ enjoy the property described in the criterion. The solution to this problem is highly nontrivial. If S consists only of one particle, the representation associated with it has to be irreducible. In this case we know that it can be interpreted as being induced on $P_+^\uparrow$ from a certain proper subgroup. There is a theorem of Mackey's which gives the decomposition of the restriction of an induced representation to a subgroup, the so-called subgroup theorem (cf. MACKEY [14], BARUT & RĄCZKA [2]). Applying this theorem one arrives at the following result: A relativistic particle is localizable in $\mathbb{R}^3$ if and only if it has real nonzero mass, or if its mass and spin are both zero. Among other things, it follows from this observation that a single photon will not be localizable (at least not in $\mathbb{R}^3$). For a proof, see WIGHTMAN [22] or BARUT & RĄCZKA [2, prop. 20.1].

It is possible to change axiom V in such a way as to enable localization of massless particles with nonzero spin and of particles of imaginary mass, in other homogeneous spaces of E(3) then $\mathbb{R}^3$. (Cf. BARUT & RĄCZKA [2, ch.20]).

The above described method can also be applied to nonrelativistic systems in space-time. Then the Poincaré group has to be replaced by the Galilei group. For details, see WIGHTMAN [22].

Next we will show how to derive a set of position operators for the system S, from the projection-valued measure E. For each ψ in the space of states H, we define a finite, positive Borel measure μ_ψ on $\mathbb{R}^3$, by

$$\mu_\psi(B) := (\psi, E(B)\psi), \qquad B \in \mathcal{B}(\mathbb{R}^3).$$

By means of these measures, we can define three generally unbounded operators Q_i, $i = 1,2,3$, on H, by

$$(\psi, Q_i\psi) := \int_{\mathbb{R}^3} x_i \, d\mu_\psi(x).$$

It can be verified straightforwardly, that these definitions are legitimate, and, moreover, that the Q_i are self-adjoint. By virtue of the axioms II and III, we have

$$[Q_i, Q_j] = 0, \qquad 1 \leq i,j \leq 3, \tag{3.13}$$

i.e., the operators Q_i commute with each other. We denote them symbolically by

$$Q_i = \int_{\mathbb{R}^3} x_i \, dE(x).$$

Let (y,R) denote an element of E(3), with $y \in \mathbb{R}^3$ and $R \in SO(3)$, such that

$$(y,R)[x] = R(x) + y, \qquad x \in \mathbb{R}^3$$

and set $R = (r_{ij})$, $R^{-1} = (r_{ij}^{-1})$. Then we have

$$\begin{aligned}
V(y,R)Q_iV(y,R)^{-1} &= \int_{\mathbb{R}^3} x_i d(V(y,R)E(x)V(y,R)^{-1}) \\
&= \int_{\mathbb{R}^3} x_i dE((y,R)[x]) = \int_{\mathbb{R}^3} ((y,R)^{-1}[x])_i dE(x) \\
&= \int_{\mathbb{R}^3} ((-R^{-1}(y),R^{-1})[x])_i dE(x) \\
&= \int_{\mathbb{R}^3} \sum_j r_{ij}^{-1}(x_j - y_j) dE(x) \\
&= \sum_j r_{ij}^{-1}\left[\int_{\mathbb{R}^3} x_j dE(x) - \int_{\mathbb{R}^3} y_j dE(x)\right] \\
&= \sum_j r_{ij}^{-1}(Q_j - y_j I).
\end{aligned}$$

This identity expresses the transformation property of the "position vector" (Q_1,Q_2,Q_3) under symmetries implied by E(3). It comes up to the expectations one would have from a rightly defined set of position operators. Moreover, from this identity we can derive the Heisenberg commutation relations. Indeed, by a theorem of Stone there exist three commuting selfadjoint operators P_k, $k = 1,2,3$, on $\mathcal{H}$ such that

$$V(y,I) = \exp i(y_1P_1 + y_2P_2 + y_3P_3), \qquad \forall y \in \mathbb{R}^3 .$$

These operators are called the momentum operators of S; they are the generators of the three-parameter translation group

$$\{V(y,I);\ y \in \mathbb{R}^3\}.$$

From the transformation rule given above for the three vector (Q_1,Q_2,Q_3) one derives readily (for instance, by formal differentiation) the following relations:

$$[Q_j, P_k] = -i\delta_{jk} I, \qquad 1 \leq j,k \leq 3.$$

Together with (3.13)

$$[P_j, P_k] = 0, \qquad 1 \leq j,k \leq 3,$$

we have the Heisenberg commutation relations, which therefore fit perfectly in our model.

Here we arrive at the more general problem of finding the representations of an algebra generated by 2n formal elements $Q_1,\ldots,Q_n$, $P_1,\ldots,P_n$, which satisfy

$$\left.\begin{array}{l} [P_j,P_k] = [Q_j,Q_k] = 0 \\ \\ [Q_j,P_k] = -i\delta_{jk} I \end{array}\right\} \quad i \leq j,k \leq n.$$

This problem can be solved in a very rigorous (and nice) way by application of the imprimitivity theorem (or, rather, a corollary of the imprimitivity theorem). Indeed, it can be shown that the well-known Schrödinger representation on $L^2(\mathbb{R}^n)$, defined by

$$\left.\begin{array}{l} (Q_j f)(x) = x_j f(x) \\ \\ (P_j f)(x) = -i \left.\dfrac{\partial f}{\partial x_j}\right|_x \end{array}\right\} \quad f \in L^2(\mathbb{R}^n),$$

is in a sense unique. This can be found in, for instance, MACKEY [14]; JAUCH [8]; BARUT & RĄCZKA [2].

<u>REMARK</u>. After the proof of theorem 3.6 we mentioned that the notions of induction and imprimitivity can be generalized to projective representations such as to enable an extension of the imprimitivity theoren to these representations. This can be used in the case that the repreresentation U is projective, and the procedure described above can be followed without any substantial modification.

Appendix to §3.1.

In this appendix we intend to finish the proof of theorem 3.4. For this purpose, it is convenient to introduce a certain dense subspace of the representation space of a unitary representation of a lcsc. group, the so-called Gårding space of a representation.

Let τ be a unitary representation of a lcsc. group G. Then τ defines a nondegenerate representation $\tilde{\tau}$ of the convolution algebra $L^1(G)$, by

$$\text{(A.1)} \qquad \tilde{\tau}(\alpha) := \int_G \alpha(x)\tau(x)dx, \qquad \alpha \in L^1(G),$$

(cf. KIRILLOV [9, 10.2]). We define a subspace $\mathcal{D}_\tau$ of the representation space $\mathcal{H}(\tau)$, by

$$\mathcal{D}_\tau := \text{linear span of } \{\tilde{\tau}\ \alpha(\xi),\ \alpha \in \mathcal{K}(G),\ \ \xi \in \mathcal{H}(\tau)\}.$$

(Note that $\mathcal{H}(\tau) = \mathcal{H}(\tilde{\tau})$.) By virtue of (A.1), we have

(A.2)(i) $\quad \tau(x)\mathcal{D}_\tau \subset \mathcal{D}_\tau, \qquad \forall x \in G;$

(A.2)(ii) $\quad T\mathcal{D}_\tau \subset \mathcal{D}_\tau, \qquad \forall T \in I(\tau).$

From the fact that $\tilde{\tau}$ is nondegenerate (i.e. $\tilde{\tau}(L^1(G))$ has trivial null space), together with the fact that $\mathcal{K}(G)$ lies dense in $L^1(G)$, it follows that

(A.2)(iii) $\quad \mathcal{D}_\tau$ lies dense in $\mathcal{H}(\tau)$.

We will call $\mathcal{D}_\tau$ the *Gårding space* of the representation τ.

Next, suppose that τ is induced by a representation σ of a closed subgroup H of G. Then the corresponding representation of $L^1(G)$ is given by

$$(\tilde{\tau}(\alpha)f)(y) = \int_G \alpha(x)f(x^{-1}y)R(\bar{y},x^{-1})^{\frac{1}{2}}dx, \quad y \in G, \alpha \in L^1(G), f \in \mathcal{H}(\tau),$$

where R is a continuous strictly positive R-function corresponding to a quasi-invariant measure on G/H. It can be shown that in this case we have

(A.2)(iv) each function in $\mathcal{D}_\tau$ is continuous;

(A.2)(v) $\{f(e);\ f \in \mathcal{D}_\tau\}$ lies dense in $\mathcal{H}(\sigma)$.

(cf. BARUT & RĄCZKA [2, ch.16].)

<u>Proof of theorem 3.4 (continued)</u>. Let σ be a unitary representation of a closed subgroup H of a lcsc. group G, write $\tau = \sigma^G$, and let $(G/H,\tau,P)$ denote the canonical system associated with τ. We had defined a mapping $T \to \hat{T}$ from $I(\sigma)$ into $L(\mathcal{H}(\tau))$ by

$$(\hat{T}f)(x) := Tf(x), \qquad f \in \mathcal{H}(\tau),$$

cf. formula (3.6). In the first part of the proof of theorem 3.4 we showed that this mapping is injective and that its range lies in $I(\tau,P)$, the commuting algebra of the canonical system. We will now prove surjectivity onto $I(\tau,P)$.

Let S belong to $I(\tau,P)$ and let $\mathcal{D}_\tau$ be the Gårding space of τ. By (A.2)(ii), we have

(A.3) $$S\mathcal{D}_\tau \subset \mathcal{D}_\tau .$$

For each f in $\mathcal{H}(\tau)$ and each Borel set B in G/H, we have

$$\begin{aligned}\int_B \|(Sf)(x)\|^2 d\mu(\bar{x}) &= \int_{G/H} \|\chi_B(x)(Sf)(x)\|^2 d\mu(\bar{x}) = \|(P_B S)f\|^2 \\ &= \|(SP_B)f\|^2 \le \|S\|^2 \,.\, \|P_B f\|^2 \\ &= \|S\|^2 \int_B \|f(x)\|^2 d\mu(\bar{x}).\end{aligned}$$

For f in $\mathcal{D}_\tau$, this means

(A.4) $$\|(Sf)(x)\| \le \|S\|\cdot\|f(x)\|, \qquad \forall x \in G,$$

since (i) $x \to \|(Sf)(x)\|^2$ and $x \to \|f(x)\|^2$ are both continuous, by (A.2)(iv) and (A.3), and (ii) nonvoid open subsets of G/H have positive μ-measure. Consider the dense subspace $\{f(e);\ f \in \mathcal{D}_\tau\}$ of $\mathcal{H}(\sigma)$ ((A.2)(v)). We define an operator S_0 on this space by

$$S_0\colon f(e) \to (Sf)(e), \qquad f \in \mathcal{D}_\tau.$$

By virtue of (A.4), this definition is legitimate, and S_0 is bounded. Therefore, there exists a unique extension to $\mathcal{H}(\sigma)$, which we denote by S_0 as well. Since $S \in I(\tau)$, we have

$$(Sf)(x) = (R(\bar{e},x))^{-\frac{1}{2}}(\tau(x^{-1})Sf)(e) = (R(e,x))^{-\frac{1}{2}}(S\tau(x^{-1})f)(e)$$

$$= S_0(R(\bar{e},x))^{-\frac{1}{2}}(\tau(x^{-1})f)(e) = S_0 f(x), \quad \forall f \in \mathcal{D}_\tau,\ \forall x \in G.$$

Furthermore,

$$\sigma(h)S_0 f(e) = \sigma(h)(Sf)(e) = (Sf)(h^{-1}) = S_0 f(h^{-1}) = S_0\sigma(h)f(e).$$

Hence S_0 commutes with the operators $\sigma(h)$ on a dense subspace of $\mathcal{H}(\sigma)$, and by continuity we conclude $S_0 \in I(\sigma)$, which proves our theorem, since $\hat{S}_0 = S$. $\square$

IV THE REPRESENTATIONS OF SEMIDIRECT PRODUCTS

4.1. Semidirect products

Let G be a lcsc. group, and suppose that we are given two closed subgroups N and H of G, such that

(i) N is invariant;
(ii) $G = N.H$ and $G \approx N\times H$;
(iii) $N \cap H = \{e\}$.

Then we shall call G the *semidirect product* of N and H. Note that each element h of H defines an automorphism of N, by

$$(4.1) \qquad h\colon n \to h\,nh^{-1} =: \alpha_h(n), \qquad n \in N.$$

Since the group operations in G are continuous the mapping $(h,n) \to \alpha_h(n)$ is continuous on $H \times N$. The multiplication in G may be written as

$$nhmk = nhmh^{-1}\, hk = n\alpha_h(m)\, hk, \quad n,m \in N, h,k \in H.$$

Conversely, if we are given two groups N and H, such that there exists a one-to-one homomorphism $\alpha\colon h \to \alpha_h$ from H into the automorphism group of N, then we can provide the Cartesian product $N \times H$ with a group structure by defining

$$(n,h)(m,k) := (n\alpha_h(m), hk), \quad n,m \in N, h,k \in H.$$

Note that the homomorphism property of $h \to \alpha_h$ is needed in order to ensure associativity of this structure. The group obtained in this manner is called the semidirect product of N and H relative to α, and usually denoted by

$$N \circledS H.$$

It will be a lcsc. group in the product topology if N and H are lcsc. groups and if the mapping $(h,n) \to \alpha_h(n)$ is continuous.

Throughout the remainder of this subsection we will assume that G is a lcsc. group, and that $G = N \circledS H$, and, moreover, we will take N to be abelian.

Let $\hat{N}$ be the family of all irreducible characters of N, that is, N consists of all continuous homomorphism $\phi\colon N \to \mathbb{T}$. If $\phi \in \hat{N}$, then we define a new element $h[\phi]$ of N for each h in H, by

$$(4.2) \qquad (h[\phi])(n) := \phi(\alpha_{h^{-1}}(n)).$$

It is easily verified that $h[\phi]$ is indeed a member of $\hat{N}$. The character $h[\phi]$ is said to be conjugate to ϕ. Recalling the definition of a G-space (§1.5), one sees that the action of H on $\hat{N}$ defined by $h\colon\phi \mapsto h[\phi]$, makes $\hat{N}$ into a H-space. Indeed, we have

$$((h_1h_2)[\phi])(n) = \phi(\alpha_{(h_1h_2)^{-1}}(n)) = \phi(\alpha_{h_2^{-1}}\alpha_{h_1^{-1}}(n)) =$$

$$= (h_2[\phi])(\alpha_{h_1^{-1}}(n)) = (h_1[h_2[\phi]])(n).$$

This property of the H-action is , of course, the reason of introducing inversion into the definition given by formula (4.2).

It is well-known that the dual of a lcsc. group can be made into a lcsc. group in its own right, by taking multiplication of characters as composition, and providing it with the so-called topology of uniform convergence on compacta (cf. KIRILLOV [9, 7.3]). A basis for the open sets of this topology is formed by the family of sets $U(C,\varepsilon,\phi_0) \subset \hat{N}$, defined by

$$U(C,\varepsilon,\phi_0) := \{\phi\in\hat{N};\ |\phi(x)-\phi_0(x)| < \varepsilon,\ \forall x \in C\},$$

where C is a compact subset of N, ε a positive nonzero real number, and ϕ_0 an element of $\hat{N}$. The continuity of the mappings $(h,n) \to \alpha_h(n)$ and $n \to \phi(n)$ ($\forall h\in H$, $n\in N$, $\phi\in\hat{N}$) leads via the inequality

$$|(h[\phi])(x)-\phi_0(x)| \leq |\phi(\alpha_{h^{-1}}(x))-\phi_0(\alpha_{h^{-1}}(x))| +$$

$$+ |\phi_0(\alpha_{h^{-1}}(x))-\phi_0(x)|,$$

and some simple standard arguments, to the conclusion that the mapping

$$(h,\phi) \to h[\phi]$$

from $H\times\hat{N}$ into $\hat{N}$ is continuous. Hence, $\hat{N}$ is a continuous H-space. The orbit structure in $\hat{N}$ will play an important role in subsection 4.3. Let $\omega \subset \hat{N}$ be an H-orbit, and let H_0 be the stabilizer in H of a fixed point $\phi_0 \in \omega$. Then H_0 is a closed subgroup of H, and we consider the mapping

$$\beta:\ hH_0 \to h[\phi_0]$$

from the homogeneous space H/H_0 onto ω. This mapping is clearly one-to-one. Furthermore, if we consider the Borel structure on ω generated by the relative topology which ω inherits from $\hat{N}$, and the Borel structure on H/H_0 generated by the quotient topology, then it can be shown that the mappings β and β^{-1} are both Borel mappings, which is expressed by calling β a Borel isomorphism (see VARADARAJAN [21, thm.8.11]). It can also be verified, by a simple argument, that ω is a Borel subset of $\hat{N}$ (ibidem, p.12). Notice that these observations enable us to identify the Borel measures (or projection-valued measures) on $\hat{N}$, restricted to ω, with those on H/H_0.

Since $\hat{N}$ is a continuous H-space, the mapping β is continuous. However, the inverse mapping $\beta^{-1}:\omega \to H/H_0$ need not be continuous, so β is generally not a homeomorphism. It is known that a sufficient condition for β to be a homeomorphism is that ω is a lcsc. space with respect to its relative topology (cf. VARADARAJAN [21, thm.8.11]). One verifies readily that this condition is satisfied if, for instance, H_0 is compact.

The discussion of examples is postponed to the end of the treatment of the representation theory.

4.2. The representation of finite semidirect products

The author thinks that a good understanding of the representation theory of general locally compact second countable semidirect products benefits from a preliminary discussion of the finite case. For, the arguments which we will use to derive a classification of the representations of finite semidirect products can be extended to infinite groups with only standard adjustments of a measure theoretical kind. This will be shown in the next subsection.

We are aware of the fact that our treatment of the finite case is amenable to substantial simplifications and generalizations, but our strategy is attuned to the infinite case.

Thus, let G denote the finite semidirect product of an abelian invariant subgroup N and a subgroup H. Consider a representation τ of N (or any finite abelian group). It can be decomposed into a linear combination of characters.

$$\tau = \sum_{\phi\in\hat{N}}^{\oplus} n_\phi \phi,$$

where $\{n_\phi\}_{\phi\in\hat{N}}$ is a set of natural numbers, uniquely determined by τ. This decomposition corresponds to a decomposition of the representation space $\mathcal{H}(\tau)$, which is also unique:

$$\mathcal{H}(\tau) = \sum_{\phi\in\hat{N}}^{\oplus} \mathcal{H}_\phi, \qquad \mathrm{Dim}(\mathcal{H}_\phi) = n_\phi.$$

Let P_ϕ be the projection operator of $H(\tau)$ which has H_ϕ as its range. Then

$$\tau(n) = \sum_{\phi\in\hat{N}} \phi(n)\, P_\phi, \quad n \in N. \tag{4.3}$$

(Note that this decomposition is the finite counterpart of the spectral decomposition of representations of locally compact abelian groups, provided by the SNAG-theorem (§4.3)). As explained in the preceding section we can view upon $P : \phi \to P_\phi$ as a projection valued measure (based on $\hat{N}$) by setting

$$P_E := \sum_{\phi\in E} P_\phi,$$

for any subset E of $\hat{N}$.

If σ is a representation of G, then $\tau := \sigma|_N$ is a representation of N. For any representation τ of N we will denote the corresponding projection valued measure on $\hat{N}$ by $P^\tau : E \to P^\tau_E$. We can now state the following lemmata on the relationship between σ and $P^{\sigma|N}$.

LEMMA 4.1. *Let τ and ρ be representations of* N *and* H *on the same Hilbert space. Then the following assertions are equivalent:*

(i) *There exists a representation σ of* G *such that $\sigma|_N = \tau$ and $\sigma|_H = \rho$.*

(ii) *The triple $(\hat{N},\rho,P^\tau)$ is a system of imprimitivity for* H.

PROOF. (i) ⇒ (ii). Condition (i) implies (and is implied by) the following identity:

$$\rho(h)\ \tau(n)\ \rho(h)^{-1} = \tau(hnh^{-1}), \qquad \forall n \in N,\ \forall h \in H. \tag{4.4}$$

Using the decomposition (4.3) of τ, we obtain

$$\rho(h)\Big(\sum_{\phi\in\hat{N}} \phi(n)P^\tau_\phi\Big)\ \rho(h)^{-1} = \sum_{\phi\in\hat{N}} \phi(hnh^{-1})\ P^\tau_\phi .$$

The right hand side can be rewritten as:

$$\sum_{\phi\in\hat{N}} \phi(hnh^{-1})\ P^\tau_\phi = \sum_{\phi\in\hat{N}} (h^{-1}[\phi])(n)\ P^\tau_\phi = \sum_{\phi\in\hat{N}} \phi(n)\ P^\tau_{h[\phi]} .$$

Hence, we have

$$(4.5) \qquad \sum_{\phi\in\hat{N}} \phi(n)(\rho(h)P_\phi^\tau\rho(h)^{-1}) = \sum_{\phi\in\hat{N}} \phi(n)\ P^\tau_{h[\phi]}.$$

By uniqueness of decomposition it follows that

$$(4.6) \qquad \rho(h)\ P_\phi^\tau\ \rho(h)^{-1} = P^\tau_{h[\phi]}, \qquad \forall h \in H,\ \forall\phi \in \hat{N}.$$

But this implies (by linearity) that condition (ii) is satisfied, so we are through.

(ii) ⇒ (i). Obviously the above argument van be reversed, that is, (4.6) implies (4.4). But then $\sigma(nh) := \tau(n)\ \rho(h)$ is a representation of G. □

LEMMA 4.2. *Let* σ_1 *and* σ_2 *be representations of* G. *Then the intertwining space* $I(\sigma_1,\sigma_2)$ *is equal to the intertwining space* $I((\sigma_{1|H},P^{\sigma_1|N}),(\sigma_{2|H},P^{\sigma_2|N}))$ *of the corresponding (by lemma* 4.1*) imprimitivity systems for* H. *In particular, one has*

(i) $\sigma_1 \simeq \sigma_2$ *if and only if the corresponding systems are equivalent;*
(ii) *a representation of* G *is irreducible if and only if the corresponding system is irreducible.*

PROOF. It is clear that

$$I(\sigma_1,\sigma_2) = I(\sigma_{1|N},\sigma_{2|N}) \cap I(\sigma_{1|H},\sigma_{2|H}).$$

Furthermore we have $T \in I(\sigma_{1|N},\sigma_{2|N})$ iff

$$T\sigma_1(n) = \sigma_2(n)T, \qquad \forall n \in N$$

iff

$$\sum_{\phi\in\hat{N}} \phi(n)\ TP_\phi^{\sigma_1|N} = \sum_{\phi\in\hat{N}} \phi(n)\ P_\phi^{\sigma_2|N}\ T, \qquad \forall n \in N,$$

iff, for all $\xi \in \mathcal{H}(\sigma_1)$ and $\eta \in \mathcal{H}(\sigma_2)$:

$$\sum_{\phi\in\hat{N}} \phi(n)(TP_\phi^{\sigma_1|N}\xi,\eta) = \sum_{\phi\in\hat{N}} \phi(n)(P_\phi^{\sigma_2|N}T\xi,\eta), \qquad \forall n \in N.$$

Since the elements of $\hat{N}$ form an orthonormal basis for the space of all complex-valued functions on N (cf. §1.2), the last identity is equivalent to

$$(TP_{\phi}^{\sigma_1|N}\xi,\eta) = (P_{\phi}^{\sigma_2|N}T\xi,\eta), \ \forall\xi \in H(\sigma_1),\ \forall\eta \in H(\sigma_2),\ \forall\phi \in \hat{N}.$$

But this is true if and only if

$$TP_E^{\sigma_1|N} = P_E^{\sigma_2|N}\,T, \qquad \forall E \subset \hat{N}.$$

Therefore, we have

$$T \in I(\sigma_1|_N,\sigma_2|_N) \Longleftrightarrow T \in I(P^{\sigma_1|N},P^{\sigma_2|N}),$$

which implies

$$I(\sigma_1,\sigma_2) = I(P^{\sigma_1|N},P^{\sigma_2|N}) \cap I(\sigma_1|_N,\sigma_2|_H) = I((\sigma_1|_H,P^{\sigma_1|N}),(\sigma_2|_H,P^{\sigma_2|N})).$$

This proves the first statement of the lemma. (i) and (ii) are immediate consequences. □

Next we show how to construct a number of irreducible representations of G.

Fix a point ϕ_0 in $\hat{N}$, and let ω_0 denote the orbit of ϕ_0 in $\hat{N}$ under the action of H, i.e.,

$$\omega_0 := \{h[\phi_0];h \in H\}.$$

Then ω_0 is H-homeomorphic with H/H_0, where

$$H_0 := \{h \in H;h[\phi_0] = \phi_0\}$$

denotes the stabilizer in H at ϕ_0.

Now, let ρ be an irreducible representation of H_0, and, for each element nh of G, define an operator $\sigma(nh)$ in the induced representation space $H(\rho^H)$ by

$$(\sigma(nh)f)(x) := (x[\phi_0])(n)(\rho^H(h)f)(x) = (x[\phi_0])(n)f(h^{-1}x),\ x \in H.$$

It is obvious that $\sigma(nh)f$ does belong to $H(\rho^H)$. We show that σ is a representation of G:

$$\begin{aligned}(\sigma(nh)\sigma(mk)f)(x) &= (x[\phi_0])(n)(\sigma(mk)f)(h^{-1}x) = \\ &= (x[\phi_0])(n)(h^{-1}x[\phi_0])(m)f((hk)^{-1}x) = \\ &= \phi_0(x^{-1}nx)\phi_0(x^{-1}hmh^{-1}x)f((hk)^{-1}x) = \\ &= \phi_0(x^{-1}nhmh^{-1}x)f((hk)^{-1}x) = \\ &= (x[\phi_0])(nhmh^{-1})f((hk)^{-1}x) = \\ &= (\sigma(nhmh^{-1}hk)f)(x) = \\ &= (\sigma(nhmk)f)(x).\end{aligned}$$

Let $\tau := \sigma_{|N}$. By virtue of lemma 4.1, $(\hat{N},\sigma_{|H},P)$ is a s.o.i. for H, and lemma 4.2(i) implies that σ is determined up to equivalence by this system. We now determine P^τ:

$$(\tau(n)f)(x) = (x[\phi_0])(n)f(x) = (\sum_{y\in H/H_0} (y[\phi_0])(n)\ \chi_{\{yH_0\}}\cdot f)(x).$$

The second step is legitimate since $x \in H$ belongs to exactly one H_0-coset, say y_0H_0, and then $y_0[\phi_0] = x[\phi_0]$, since H_0 stabilizes ϕ_0. Next we define a projection-valued measure based on $\hat{N}$ and acting in $H(\rho^h)$, by

$$P_E := \sum_{\phi\in E} P_\phi,\quad E \subset \hat{N},$$

where

$$(4.7)\qquad (P_\phi(f))(x) := \begin{cases}0 & \text{if } \phi \in \hat{N}\backslash\omega_0\\ \chi_{\{yH_0\}}(x)\cdot f(x) & \text{if } \phi = y[\phi_0],\ y \in H.\end{cases}$$

Then we may write

$$(\tau(n)f)(x) = (\sum_{\phi\in\hat{N}} \phi(n)P_\phi(f))(x),\qquad f \in H(\rho^H),$$

so P is the projection-valued measure associated with τ. Moreover, P is based on ω_0, actually, since it vanishes on $N\backslash\omega_0$. We express this fact by saying that P is *concentrated in one orbit* (ω_0). Besides, we know that ω_0 is H-homeomorphic with H/H_0, and that the homeomorphism is given by

$$\Phi : y\,H_0 \to y[y_0].$$

Consequently, we may consider P as a projection-valued measure on H/H_0, by defining:

$$P_E := P_{\Phi(E)}, \quad E \subset H/H_0.$$

Then we find (by (4.7)):

$$(4.8) \qquad (P_E(f))(x) = (P_{\Phi(E)}(f))(x) = \chi_E(\bar{x})f(x), \qquad E \subset H/H_0,\ f \in \mathcal{H}(\rho^H).$$

But formula (4.8) defines a projection-valued measure equal to the one occuring in the canonical imprimitivity system of ρ^H (cf. example 3.3). Hence, by corollary 3.5, the irreducibility of ρ implies irreducibility of the system $(N,\sigma_{|H},P^\tau) = (H/H_0,\rho^H,P)$ (note that $\sigma_{|H} = \rho^H$ by definition), which in its turn results in irreducibility of σ, by virtue of lemma 4.2(ii). Finally, lemma 4.2(i) together with the imprimitivity theorem yields that σ is determined up to equivalence by ρ.

In the construction of σ we have chosen a fixed point ϕ_0 in ω_0, but it will turn out in the next theorem that the collection of representations, obtained by letting ρ run through $\hat{H}_0$ is independent (up to equivalence) of the choice of ϕ_0 in ω_0. This fact can also easily be verified straightforwardly.

We shall call the above constructed representation σ of G associated with the orbit ω_0 and the representation $\rho \in \hat{H}_0$, and denote it by $\sigma^{(\omega_0,\rho)}$. The following theorem concludes the discussion of representations of finite semi-direct products.

<u>THEOREM 4.3</u>. (Mackey) *Let* $\{\omega\}_{\omega\in\Omega}$ *be the collection of* H*-orbits in* $\hat{N}$ *and let* H_ω *be the stabilizer in* H *at a fixed point of* ω. *Then:*

(i) $\sigma^{(\omega,\rho)}$ *is an irreducible representation of* G *for all pairs* (ω,ρ), *in which* ρ *is an irreducible representation of* H_ω;

(ii) $\sigma^{(\omega,\rho)} \simeq \sigma^{(\omega',\rho')}$ *if and only if* $\omega = \omega'$ *and* $\rho \simeq \rho'$;

(iii) *Each member of* $\hat{G}$ *is of the form* $\sigma^{(\omega,\rho)}$, *for some* $\omega \in \Omega$ *and* $\rho \in \hat{H}_\omega$.

PROOF. (i) was already proved above. There we also showed that $\sigma^{(\omega,\rho)} \simeq \sigma^{(\omega,\rho')}$ if and only if $\rho \simeq \rho'$. As to the role of the orbit ω in determining the equivalence class of $\sigma^{(\omega,\rho)}$, it suffices to make the obvious observation that the restrictions to N of $\sigma^{(\omega,\rho)}$ and $\sigma^{(\omega',\rho')}$ will not be equivalent if $\omega \neq \omega'$. This proves (ii).

(iii): Let σ be an irreducible representation of G, and consider the projection-valued measure $P := P^{\sigma|N}$. We contend that P is concentrated on one orbit. Indeed, let ω be an orbit, then by virtue of the identity

$$\sigma|_H(h)P_\omega\, \sigma|_H(h)^{-1} = P_{h[\omega]} = P_\omega, \qquad \forall h \in H,$$

it follows that P_ω commutes with all operators $\sigma|_H(h)$, $h \in H$. Furthermore, P_ω commutes with $\sigma|_N$ as well, so $P_\omega = 0$ or I, by the irreducibility of σ.

Suppose that $P_\omega = 0$ for all orbits ω. Then we would have

$$I = P_{\hat{N}} = \sum_{\omega\in\Omega} P_\omega = 0;$$

a contradiction. On the other hand, it is obvious that

$$P_{\omega_1} = P_{\omega_2} = I$$

is not allowed, unless $\omega_1 = \omega_2$. Hence, there is exactly one orbit, say ω_0, with $P_{\omega_0} = I$ and $P_{\hat{N}\backslash\omega_0} = 0$. (*).

But then we may view P as a projection-valued measure on $H/H_{\omega_0} \approx \omega_0$, and therefore we see that $(H/H_{\omega_0}, \sigma|_H, P^{\sigma|N})$ is a transitive s.o.i. for H (lemma 4.1). By the imprimitivity theorem it follows that:

(i) $\sigma|_H$ is induced on H by a certain representation ρ of H_{ω_0} and
(ii) the system is equivalent to the canonical system of $\rho^{H^{\omega_0}}$.

Since σ is irreducible, ρ is also irreducible, by lemma 4.2(ii) and corollary 3.5, and this fact together with lemma 4.2(i) yields that σ is equivalent to $\sigma^{(\omega_0,\rho)}$. □

(*): This paragraph marks the main difference between the representation theories of finite and general lcsc. semi-direct products. In fact, let P be a projection-valued measure based on a continuous G-space, where G is a lcsc. group. Then $P_\omega = 0$ for each orbit ω does not necessarily imply that $\sum_{\omega\in\Omega} P_\omega = 0$, since this "sum" may be continuous. By laying a condition of a measure theoretical kind on the orbit structure of the G-space, this defect can be repaired. This will be shown in the next subsection.

EXAMPLE 2.4. In §1.2 we discussed the permutation group S_3. Let $N = A_3$, the alternating subgroup and $H = \{(1),(12)\}$, a cyclic subgroup of order 2. Then it is readily verified that S_3 is the semi-direct product of N and H. The characters of A_3 were denoted by ψ_1, ψ_2, ψ_3 in example 1.11. The group H acts on A_3 by

$$(1)[\psi_i] = \psi_i, \qquad i = 1,2,3, \qquad \text{and}$$

$$(12)[\psi_1] = \psi_1; \qquad (12)[\psi_2] = \psi_3; \qquad (12)[\psi_3] = \psi_2.$$

Hence there are two orbits:

$$\omega_1 = \{\psi_1\}, \qquad \text{stabilizer: } H_1 = H;$$
$$\omega_2 = \{\psi_2,\psi_3\}, \qquad \text{stabilizer: } H_2 = \{(1)\}.$$

Note that the character table of H is

	(1)	(12)
ρ_1	1	1
ρ_2	1	-1

By some elementary computations we find:

$$\sigma^{(\omega_1,\rho_1)} = \chi_1; \quad \sigma^{(\omega_1,\rho_2)} = \chi_2; \quad \sigma^{(\omega_2,1)} \cong \tau_3,$$

where we use the notation of example 1.11.

In SERRE [19] the reader can find a shorter proof of theorem 4.3. In REYES [16] an analogous approach to finite semi-direct products $G = N \circledS H$ with N not abelian, is given, which proceeds by admitting irreducible projective representations of the little groups in constructing the irreducible representations of G.

4.3. The representations of lcsc. semi-direct products

Throughout this subsection G denotes a lcsc. group, which is the semi-direct product of two of its subgroups N and H, with N abelian and invariant.

First we recall the statements of the SNAG theorem (cf. BARUT & RĄCZKA [2, 6.2]).

(i) *If* τ *is a unitary representation of a lcsc. abelian group* A, *then there exists a unique projection-valued measure* $P: E \mapsto P_E$, *based on* $\hat{A}$ *and acting in the representation space of* t, *such that*

$$(\tau(a)\xi,\eta) = \int_{\hat{A}} \phi(a)d\mu_{\xi,\eta}(\phi), \qquad \forall\xi,\eta \in H(\tau), \quad \forall a \in A,$$

where the complex Borel measure $\mu_{\xi,\eta}$ *on* $\hat{A}$ *is defined by*

$$\mu_{\xi,\eta}(E) := (P_E\xi,\eta), \qquad E \text{ Borel set in } \hat{A}.$$

We write as usual:

$$\tau(a) = \int_{\hat{A}} \phi(a)dP_\phi, \qquad a \in A. \tag{4.9}$$

(ii) *Conversely, if* $P: E \to P_E$ *is a projection-valued measure on* $\hat{A}$, *acting in a certain separable Hilbert space* H, *then* (4.9) *define a unitary representation of* A *on* H.

If τ is a representation of N, then we denote by P^{τ} the projection-valued measure which corresponds to τ by virtue of this theorem.

Let τ and ρ be unitary representations of N and H respectively, with the same representation space. Suppose that they satisfy

$$\rho(h)\tau(n)\rho(h)^{-1} = \tau(hnh^{-1}), \quad \forall n \in N, \forall h \in H.$$

By decomposing τ on both sides as in (4.9), we obtain

$$\rho(h)\left(\int_{\hat{N}} \phi(n)dP^{\tau}_{\phi}\right)\rho(h)^{-1} = \int_{\hat{N}} \phi(hnh^{-1})dP^{\tau}_{\phi}.$$

This yields

$$\int_{\hat{N}} \phi(n)d(\rho(h)P^{\tau}_{\phi}\rho(h)^{-1}) = \int_{\hat{N}} (h^{-1}[\phi])(n)dP^{\tau}_{\phi} = \int_{\hat{N}} \phi(n)dP^{\tau}_{h[\phi]}.$$

This formula is obviously the infinite counterpart of formula (4.5). By the uniqueness granted in statement (i) above, we conclude that

$$\rho(h)P^{\tau}_{E}\rho(h)^{-1} = P^{\tau}_{h[E]},$$

for all h in H and all Borel subsets E of $\hat{N}$. This proves lemma 4.1 in the case of lcsc. groups.

Next, let σ_1 and σ_2 be unitary representations of G, and set $\tau_1 = \sigma_1|_N$ and $\tau_2 = \sigma_2|_N$. For T in $I(\tau_1,\tau_2)$, we have

$$(\tau_1(x)\xi,T^*\eta) = (\tau_2(x)T\xi,\eta), \quad \forall\xi \in H(\sigma_1), \quad \forall\eta \in H(\sigma_2), \quad \forall x \in N.$$

Hence

$$\text{(4.10)} \qquad \int_{\hat{N}} \phi(n)d\mu^{\tau_1}_{\xi,T^*\eta}(\phi) = \int_{\hat{N}} \phi(n)d\mu^{\tau_2}_{T\xi,\eta}(\phi), \qquad \forall n \in N.$$

The following result can be proved (see appendix):

- *If* μ *is a finite complex-valued measure on the dual* $\hat{A}$ *of a lcsc. abelian group* A, *such that*

$$\int_{\hat{A}} \phi(a)d\mu(\phi) = 0, \qquad \forall a \in A,$$

then μ *is equal to zero.*

Hence, we have

$$\mu^{\tau_1}_{\xi,T^*\eta} = \mu^{\tau_2}_{T\xi,\eta}, \qquad \forall\xi \in H(\sigma_1), \quad \forall\eta \in H(\sigma_2).$$

This implies

$$(P^{\tau_1}\xi,T^*\eta) = (P^{\tau_2}T\xi,\eta), \qquad \forall\xi \in H(\sigma_1), \quad \forall\eta \in H(\sigma_2).$$

Consequently $T \in I(P^{\tau_1},P^{\tau_2})$, which proves lemma 4.2 in the case of lcsc. groups. For, it is obvious that the argument above can be reversed, whence $I(P^{\tau_1},P^{\tau_2}) = I(\tau_1,\tau_2)$.

We will now repeat the construction of representations of G. Fix an orbit ω_0 in $\hat{N}$, a point ϕ_0 in ω_0, and denote by H_0 the stabilizer in H at ϕ_0. Let ρ be an irreducible unitary representation of H_0, and let f be a function in the induced representation space $H(\rho^H)$, considered as a space of L^2-functions. For each element $nh \in G$ we define a new function $\sigma(nh)f$ on H by

$$(\sigma(nh)f)(x) := (x[\phi_0])(n)(\rho^H(h)f)(x).$$

We have

(i) $(\sigma(nh)f)(xh_0) = \rho(h_0^{-1})(\sigma(nh)f)(x), \quad h_0 \in H_0;$

(ii) $x \longmapsto ((\sigma(nh)f)(x),\xi) = (x[\phi_0])(n)((\rho^H(h)f)(x),\xi)$
is a Borel function for each $\xi \in H(\rho)$, since it is the product of two Borel functions.

(iii) $$\int_{H/H_0} \|(x[\phi_0])(n)(\rho^H(h)f)(x)\|^2 \, d\mu(\bar{x}) = \int_{H/H_0} \|(\rho^H(h)f)(x)\|^2 d\mu(\bar{x}) = \|f\|^2.$$

These properties imply that $\sigma(nh)f$ belongs to $\mathcal{H}(\rho^H)$ (cf. §2.4), and (iii) implies also $\|\sigma(nh)f\| = \|f\|$. Furthermore, we have

$$(\sigma(nh)\sigma(mk)f)(x) = (x[\phi_0])(n)(\sigma(mk)f)(h^{-1}x)(R(\bar{x},h^{-1}))^{\frac{1}{2}} =$$

$$= (x[\phi_0])(n)(h^{-1}x[\phi_0])(m)f((hk)^{-1}x)(R(\bar{x},h^{-1})(R(\overline{h^{-1}x},k^{-1}))^{\frac{1}{2}}$$

$$= (x[\phi_0])(n)(nhmk^{-1})f((hk)^{-1}x)(R(\bar{x},(hk)^{-1}))^{\frac{1}{2}} = (\sigma(nhmh^{-1}hk)f)(x)$$

$$= (\sigma(nhmk)f)(x), \qquad n,m \in N,\ h,k \in H.$$

Here R is a continuous real function on $H/H_0 \times H$ corresponding to the quasi-invariant measure μ on H/H_0.

Putting the pieces together, we see that σ is a homomorphism from G into the algebra of unitary operators on $\mathcal{H}(\rho^H)$. For proving weak continuity of σ, it is sufficient to do so on the dense subspace K_ρ of $\mathcal{H}(\rho^H)$. If $f,g \in K_\rho$, then the function

$$nh \to \int_{H/H_0} (x[\phi_0])(n)((\rho^H(h)f)(x),g(x))d\mu(\bar{x}),$$

can be easily showed to be continuous, by standard arguments.

We conclude that σ is a unitary representation of G. It will be called associated with the orbit ω_0 and the representation ρ, and denoted by $\sigma^{(\omega_0,\rho)}$.

If we set $\tau = \sigma|_N$, then

$$(\tau(n)f)(x) = (x[\phi_0])(n)f(x), \qquad f \in \mathcal{H}(\rho^H).$$

We contend that this identity implies that the projection-valued measure P^τ on $\hat{N}$, associated with τ, is concentrated on the orbit ω_0. To prove this, we define a projection-valued measure $P\colon E \to P_E$ on $\hat{N}$, which acts in $\mathcal{H}(\rho^H)$, by

$$(P_Ef)(x) = \chi_E(x[\phi_0])f(x), \qquad E \text{ a Borel set in } \hat{N}.$$

For each $f,g \in H(\rho^H)$, P yields a complex Borel measure on $\hat{N}$:

$$\mu_{f,g}(E) = (P_E f,g) = \int_{H/H_0} \chi_E(x[\phi_0])(f(x),g(x))d\mu(\bar{x}).$$

This can be rewritten as (by abuse of notation)

$$d\mu_{f,g}(\phi) = \begin{cases} 0 & \text{if } \phi \in \hat{N}\setminus\omega_0 \\ (f(x),g(x))d\mu(\bar{x}) & \text{if } \phi = x[\phi_0]. \end{cases}$$

Hence we find

$$\begin{aligned}(\tau(n)f,g) &= \int_{H/H_0} (x[\phi_0])(n)(f(x),g(x))d\mu(\bar{x}) \\ &= \int_{\hat{N}} \phi(n)d\mu_{f,g}(\phi),\end{aligned}$$

which proves our assertion, by virtue of the uniqueness of the projection-valued measure associated with τ.

The set of representations σ obtained by letting ρ run through (stab(ϕ_0))is independent (up to equivalence) of the choice of ϕ_0. Verification of this assertion can be done by straightforward manipulation of the definition of induced representations.

Before we state the analogue of theorem 4.3, we have to consider what happens to the third statement of this theorem. As we have pointed out after the proof of this theorem, one of the arguments used in proving the third statement does not apply to general lcsc. semi-direct products. Besides, it is possible to give counterexamples of lcsc. semi-direct products having a lot of irreducible unitary representations which can not be constructed in the above described manner. Hence we must look for a more restricted class of groups, such that theorem 4.3 carries over completely.

DEFINITION 4.5. A continuous G-space X for a lcsc. group G is said to be *countably separated* (or to have a *smooth orbit structure*) if there exists a countable sequence $B_1,B_2,...$ of Borel subsets of X, such that

$$(4.11)\quad \begin{cases} \text{(i)} & \text{each } B_i \text{ is a union of G-orbits;} \\ \text{(ii)} & \text{each orbit in X is the intersection of those } B_i \text{ that contain it.} \end{cases}$$

If G is a lcsc. semi-direct product of N and H such that the H-orbit structure in $\hat{N}$ is countably separated, then we shall call G a *regular semi-direct product*.

DEFINITION 4.6. Let P be a projection valued measure on a continuous G-space X for a lcsc. group G. Then we say that P is *almost transitive* if

$$P_\omega = 0 \text{ or } I, \quad \text{for each orbit } \omega.$$

Note that this definition implies two possibilities: $P_\omega = 0$ for all orbits; or $P_\omega = I$ and $P_{X\setminus\omega} = 0$, for a certain orbit. In the last case we call P *transitive*, or *concentrated in one orbit.*

LEMMA 4.7. *Let* P *be a projection valued measure on a continuous* G-*space with countably separated orbit structure. If* P *is almost transitive, then it is concentrated on one orbit.*

PROOF. Let $\{B_i\}_{i=1}^\infty$ be a sequence of Borel subsets of X, satisfying (4.11). Suppose $P_\omega = 0$, for all orbits ω in X. For a fixed orbit ω_0 there exists a subsequence $\{B_{n_i}\}_{i=1}^\infty$ of $\{B_i\}_{i=1}^\infty$ such that

$$\omega_0 = \bigcap_{i=1}^\infty B_{n_i}, \quad \text{and } B_{n_{i+1}} \subset B_{n_i} \qquad \text{for } i = 1,2,\ldots .$$

(Where we assume, without damaging generality, that $\{B_i\}_{i=1}^\infty$ is closed under finite intersection). But then we have

$$0 = P_{\omega_0} = P_{\bigcap_{i=1}^\infty B_{n_i}} = \lim_{i\to\infty} P_{B_{n_i}} .$$

Since each B_i is a union of orbits, $P_{B_i} = 0$ or I, for all i. Hence, the above identity implies that $P_{B_{n_i}} = 0$ for at least one B_{n_i}. Consequently, each orbit in X is contained in a Borel set B_i of P-measure zero, which in turn implies that X can be covered with a countable family of P-null-sets. Thus $P_X = 0$; a contradiction. □

For the sake of completeness we will show by means of an example that the condition of countable separateness can not be omitted in lemma 4.7.

<u>EXAMPLE 4.8</u>. Let $\mathbb{T}$ be the circle group, consisting of all complex numbers of modulus one, and let $\mathbb{Z}$ be the additive group of integers.
We make $\mathbb{T}$ into a continuous $\mathbb{Z}$-space by defining a $\mathbb{Z}$-action on $\mathbb{T}$ by

$$n(z) = e^{in}z, \qquad n \in \mathbb{Z}, \quad z \in \mathbb{T}.$$

Consider the projection valued measure P on T, which is canonically associated with the regular representation of $\mathbb{T}$ on $L^2(\mathbb{T},\alpha)$, where α is the normalized rotation invaraint measure on $\mathbb{T}$. That is, for each Borel subset E of $\mathbb{T}$, we have

$$(P_E(f))(z) = \chi_E(z)f(z), \qquad f \in L^2(\mathbb{T},\alpha).$$

The $\mathbb{Z}$-orbits in $\mathbb{T}$ are countable, whence they have α-measure zero. If ω is an orbit, it follows that χ_ω is α-almost everywhere zero on T. Hence $P_\omega = 0$ for all orbits ω.

<u>REMARK</u>. In §4.1 we mentioned that the H-orbits in $\hat{N}$ can be provided with two topologies, the quotient topology from H/H_ω and the relative topology from $\hat{N}$. The one-to-one mapping $xH_\omega \to x[\phi]$ (where H_ω stabilizes $\phi \in \omega$) is a homeomorphism with respect to the quotient topology, and continuous with respect to the relative topology. The following highly nontrivial fact can be proved (GLIMM [4]): G is regular if and only if the mapping $xH_\omega \to x[\phi]$ is a homeomorphism with respect to the relative topology on ω from $\hat{N}$, for each orbit ω. We emphasize that in general this equivalence is only valid if G is second countable. By simple standard methods one verifies that the necessary condition is satisfied if H is compact. But, for instance the

Poincaré group is a regular semi-direct product as will be shown in subsection 4.6, and in this case H is equal to the orthochronous Lorentz group, which is not compact.

Let us assume that our semi-direct product G is regular. Then let σ be an irreducible unitary representation of G, and set $\tau = \sigma|_N$. For each orbit ω in $\hat{N}$, we see that P^τ_ω commutes with any operator belonging to one of the sets $\{\sigma(h);\ h \in H\}$ and $\{\tau(n) = \sigma(n);\ n \in N\}$. Hence, since σ is irreducible, P^τ_ω is either zero or the identity. But then, by virtue of lemma 4.7, P^τ is concentrated on one orbit, say ω_0. Therefore, we may view $(\hat{N},\sigma|_H,P^\tau)$ as a transitive system, based on H/H_0, where H_0 is the stabilizer at a fixed point of ω_0. This implies

(i) $\sigma|_H$ is induced by a certain unitary representation ρ of H_0 ;
(ii) $(H/H_0,\ \sigma|_H,\ P^\tau)$ is equivalent to the canonical system of ρ^H;

By virtue of lemma 4.2(ii) for lcsc. semi-direct products and corollary 3.5, we conclude that ρ is irreducible. Finally, from lemma 4.2(i) it follows that σ is equivalent to $\sigma^{(\omega_0,\rho)}$.

<u>THEOREM 4.9</u>. (MACKEY) *Let* G *be a lcsc. semi-direct product of* N *and* H, *with* N *abelian, let* $\{\omega\}_{\omega\in\Omega}$ *be the collection of* H*-orbits in* $\hat{N}$*, and let* H_ω *denote the stabilizer in* H *at a fixed point of* $\omega \in \Omega$*. Then, one has*

(i) $\sigma^{(\omega,\rho)}$ *is irreducible for all* ω *in* Ω *and all* ρ *in* $\hat{H}_\omega$*;*
(ii) $\sigma^{(\omega,\rho)} \simeq \sigma^{(\omega',\rho')}$ *if and only if* $\omega = \omega'$ *and* $\rho \simeq \rho'$.

If G *is regular, then:*

(iii) *The representations* $\sigma^{(\omega,\rho)}$, $\omega \in \Omega$, $\rho \in \hat{H}_\omega$, *exhaust the set of all unitary irreducible representations of* G*, up to equivalence.* □

<u>REMARK 4.10</u>. Several authors use a somewhat different construction of the representations $\sigma^{(\omega,\rho)}$ (MACKEY, LIPSMAN). They proceed as follows. Choose an element ϕ of ω, let ρ be an irreducible unitary representation of H_ω, and set

$$\tau(nh) = \phi(n)\rho(h), \qquad n \in N,\ h \in H_\omega.$$

It is readily verified that τ defines a unitary representation of the subgroup $N \circledS H_\omega$ of G. The next step is induction of τ on G, and it can be shown that τ^G is irreducible.

We suggest that the reader thinks out for himself how equivalence of τ^G and $\sigma^{(\omega,\rho)}$ can be proved.

REMARK 4.11. For convenience we wish to mention two special cases of the constructions of $\sigma^{(\omega,\rho)}$, which do often occur.

First, consider the trivial character of N, which sends all elements of N to the identity of $\mathbb{C}$. If ϕ_0 denotes this character, then its orbit is $\omega_0 = \{\phi_0\}$, and its little group comprises all of H. Hence, we get

$$\sigma^{(\omega_0,\tau)}(n,h)x = \tau(h)x, \qquad \tau \in \hat{H},\ x \in H(\tau).$$

Thus, the irreducible unitary representations associated with ω_0 are just the trivial extensions to G of the irreducible unitary representations of H. We shall call ω_0 the *trivial orbit*.

Another extreme case is the one in which the little group is trivial. Suppose that ω is an orbit with $H_\omega = \{e\}$. Then, for a fixed point ϕ of ω we obtain

$$(\sigma^{(\omega,1)}(n,h)f(x) = (x[\phi])(n)(\lambda(h)f)(x) =$$

$$= (x[\phi])(n)f(h^{-1}x), \qquad f \in L^2(H),$$

where 1 denotes the unique irreducible character of H_ω, and λ the regular representation of H on $L^2(H)$. Indeed, obviously 1^H is equivalent to λ.

We conclude this subsection with giving an idea of the large amount of literature dealing with various generalizations of theorem 2.10. In particular, one is concerned about what happens if N is no longer abelian. We give an example of the results in this state.

Let N be a closed invariant subgroup of a lcsc. group G. Then, if τ is a unitary representation of N, and if x is an element of G, the mapping $x[\tau]$ from N into $L(H(\tau))$ given by

$$x[\tau] : n \to \tau(x^{-1}nx), \qquad n \in N,$$

still defines a unitary representation of N, which will be irreducible if τ is irreducible. Hence, $\hat{N}$ can be made into a G-space. Moreover, if N is type I (cf. MACKEY [14, p.42]), then it can be shown that $\hat{N}$ is a standard Borel space (i.e. $\hat{N}$ has a Borel structure with a nice property) *), and that the mapping

$$(x,\tau) \to x[\tau]$$

is a Borel mapping from $G \times \hat{N}$ onto $\hat{N}$. Such a G-space is called a Borel G-space instead of a continuous G-space. Clearly, the definition of countable separateness extends to these spaces without alterations. If the orbit structure in $\hat{N}$ is countably separated then we say that N is *regularly embedded* in G.

We have the following theorem (MACKEY [14], LIPSMAN [10]):

<u>THEOREM 4.12</u>. *Let* N *be a type* I, *regularly embedded closed invariant subgroup of a lcsc. group* G *and denote by* G_ω *the stabilizer in* G *at a fixed point of* ω, *where* $\{\omega\}_{\omega\in\Omega}$ *is the collection of* G*-orbits in* $\hat{N}$. *Then*

$$\hat{G} = \bigcup_{\omega\in\Omega} \{\rho^G;\ \rho \in \check{G}_\omega\},$$

where, denoting by ϕ_ω *the element of* ω *stabilized by* G_ω, *the set* $\check{G}_\omega$ *is defined by*

$$\check{G}_\omega := \{\rho \in \hat{G}_\omega;\ \rho|_N \text{ is equivalent to a direct sum of } n \text{ copies of } \phi_\omega, \text{ with } n = \infty, 1, 2, \ldots\}.$$

4.4. <u>The "ax+b"-group</u>

Consider the semi-direct product G of $N = \mathbb{R}$ and $H = \mathbb{R}_+$, the multiplicative group of positive nonzero real numbers, relative to

---

*) See for instance VARADARAJAN [11, p.10].

$$\alpha_h : n \to hn, \qquad h \in \mathbb{R}_+, \qquad n \in \mathbb{R}.$$

Then $G = \{(n,h);\ n \in \mathbb{R},\ h \in \mathbb{R}_+\}$, and

$$(n,h)(m,k) = (n+hm,hk).$$

We have $\hat{N} = \mathbb{R}$, and the irreducible characters of N are given by

$$\phi_a(n) = e^{ian}, \qquad a \in \mathbb{R}.$$

H acts on $\hat{N}$ by

$$(h[\phi_a])(n) = \phi_a(\tfrac{n}{h}) = \phi_{\frac{a}{h}}(n).$$

Hence, the orbits in $\hat{N}$ are:

$$\omega_0 = \{\phi_0\}, \qquad \text{stabilizer: } H_0 = H;$$

$$\omega_+ = \{\phi_a; a>0\} \qquad \text{stabilizer: } H_+ = \{1\};$$

$$\omega_- = \{\phi_a; a<0\}, \qquad \text{stabilizer: } H_- = \{1\}.$$

Consequently, there are no proper little groups, and we find the following irreducible unitary representations of G:

<u>Ad ω_0</u>: This orbit is the trivial orbit (cf. remark 4.11); the representations associated with it are just the trivial extensions to G of the irreducible representations of H. These are given by

$$\psi_a(h) = h^{ia}, \qquad a \in \mathbb{R}.$$

Hence, we find

$$\sigma^{(\omega_0,\psi_a)}(n,h)z = h^{ia}\, z, \qquad z \in \mathbb{C}.$$

<u>Ad ω_+</u>: The little group in this case is the trivial subgroup $\{1\}$ of H. Hence, choosing ϕ_1 as a fixed point in ω_+, we obtain one irreducible unitary representation of G on $L^2(\mathbb{R}_+)$:

$$(\sigma^{(\omega_+,1)}(n,h)f)(x) = \phi_{x^{-1}}(n)f(h^{-1}x) = e^{i\frac{n}{x}} f(h^{-1}x).$$

Ad ω_-: This case is analogous to ω_+. We choose ϕ_{-1} as a fixed point, and get

$$(\sigma^{(\omega_-,1)}(n,h)f)(x) = \phi_{-x^{-1}}(n)f(h^{-1}x) = e^{-i\frac{n}{x}} f(h^{-1}x), \quad f \in L^2(\mathbb{R}_+).$$

Thus, we found a continuous family of irreducible characters, and two infinite-dimensional representations. Since the number of orbits in $\hat{N}$ is finite, G is regular and hence the above representations exhaust the set of all irreducible unitary representations of G.

The group G is usually called the "ax+b-group" (it can be interpreted as being the identity component in the group of all linear transformations of a straight line in a plane), and it is of historical interest, since it was one of the first noncompact groups to have all of its irreducible unitary representations classified (see GEL'FAND & NAIMARK [5]). Moreover, this was done before Mackey introduced his general theory.

This remark applies also to three other examples; the Euclidean groups E(2) and E(3) (§4.5) and the continuous Poincaré group $P_+^{\uparrow}$ (§4.6). The historical references for these examples are WIGNER [23] and BARGMANN [1].

4.5. The Euclidean groups

Let G be a lcsc. group, which is the semi-direct product of two of its subgroups N and H with N abelian and invariant. We say that G is a *motion group* if H is compact. Notice that this implies that G is regular (see the remark before theorem 4.9). Well-known examples of such groups are the Euclidean motion groups E(n).

Let $N = \mathbb{R}^n$ and $H = SO(n)$ (the rotation group of $\mathbb{R}^n$), and let G be the semi-direct product of N and H relative to

$$\alpha_R(x) = R(x), \qquad R \in SO(n),\ x \in \mathbb{R}^n.$$

Then G can be viewed as being the group of all rotations and translations of $\mathbb{R}^n$, and it is called the Euclidean motion group of $\mathbb{R}^n$, denoted by E(n).

The character group of N is isomorphic with $\mathbb{R}^n$, and the characters are given by

$$\phi_y(x) = e^{i(x,y)}, \qquad y \in \mathbb{R}^n,$$

where (x,y) denotes the Euclidean inner product on $\mathbb{R}^n$. For all $R \in SO(n)$, we have

$$(Rx,y) = (x,R^{-1}y), \quad \forall x,y \in \mathbb{R}^n.$$

Hence, $SO(n)$ acts on $\hat{N}$ by

$$R[\phi_y] = \phi_{R(y)}.$$

Consequently, the orbits in $\hat{N}$ are $(n-1)$-dimensional spheres, i.e.

$$\omega_r = \{\phi_x;\ (x,x) = r^2\}, \qquad r \geq 0.$$

Regularity of $E(n)$ can be verified directly. Indeed, for each ordered pair (r_1,r_2) of rational numbers with $0 < r_1 < r_2$, let a Borel set $B(r_1,r_2)$ in $\hat{N}$ be defined by

$$B(r_1,r_2) := \bigcup_{r_1<s<r_2} \omega_s,$$

and set

$$B_0 := \omega_0.$$

Then $\{B_0\} \cup \{B(r_1,r_2);\ 0 < r_1 < r_2;\ (r_1,r_2) \in \mathbb{Q}^2\}$ is a countable family of Borel subsets of $\hat{N}$, which satisfies (4.11).

The stabilizers of the fixed points $\phi_{(r,0,0,\ldots,0)}$ are given by

$H_0 := \mathrm{stab}(\phi_{(0,\ldots,0)}) = SO(n)$, and

$H_r := \mathrm{stab}(\phi_{(r,0,\ldots,0)}) = SO(n-1)$, $r < 0$.

Here we consider $SO(n-1)$ as a subgroup of $SO(n)$ by the embedding

$$R \in SO(n-1) \to \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \in SO(n).$$

In the case $n = 2$, H_0 is isomorphic with the circle group $\mathbb{T} = \{e^{i\phi};\phi \in [0,2\pi)\}$, and $H_r = \{1\}$, $r > 0$. Hence, the set of irreducible unitary representations of $E(2)$ consists of

(i) a countable family of characters, parametrized by $n = 0, \pm 1, \pm 2, \ldots$, which are the extensions to $E(2)$ of the characters of $\mathbb{T}$;

(ii) a continuous family of infinite-dimensional representations, parametrized by $r > 0$, which have the form

$$(\sigma^{(\omega_r,1)}(y,R)f)(S) = e^{ir(S^{-1}y)_1} f(R^{-1}S),$$

where S belongs to $SO(2)$, and f belongs to $L^2(SO(2),\alpha)$ with α being the rotation invariant measure on $SO(2)$.

For $n = 3$, $H_r \simeq \mathbb{T}$, $r > 0$, and $H_0 = SO(3)$. It is well-known that the set of irreducible unitary representations of $SO(3)$ consists of a countable family of representations, usually denoted by $D^{(s)}$, $s = 0,1,2,\ldots$, where the dimension of $D^{(s)}$ equals $2s+1$. Note that the unitary irreducible representations of the special unitary group $SU(2)$ are usually denoted by $D^{(s)}$ as well, for $s = 0,\frac{1}{2},1,\ldots$.

This is explained as follows: The group $SU(2)$ is the two-fold covering group of $SO(3)$, and therefore its irreducible unitary representations give rise to irreducible unitary representations of $SO(3)$, which are possibly projective with phase-factor -1. It can be shown that $D^{(s)}$ yields a proper representation of $SO(3)$ for s integer, which is also denoted by $D^{(s)}$, and a projective representation of $SO(3)$ for $s = \frac{1}{2},\frac{3}{2},\ldots$.

The set of irreducible unitary representations of $E(3)$ is given by

(i) a series $\sigma^{(s)} := \sigma^{(\omega_0,D^{(s)})}$, $s = 0,1,2,\ldots$, with $\dim(\sigma^{(s)}) = 2s+1$;

(ii) a continuous family of infinite-dimensional representations, parametrized by pairs (r,n), $r > 0$, $n = 0, \pm 1, \pm 2, \ldots$. They can be realized on the space of square integrable functions on the sphere $S^2 \approx SO(3)/SO(2)$.

4.6. The continuous Poincaré group

We start with recollecting some general facts (cf. VARADARAJAN [2, XII]).

Let $M = \mathbb{R}^4$ be the Minkowski space-time. Elements of M will be denoted by $\underline{x} = (x_0 = ct, x_1, x_2, x_3)$. The distance $(\underline{x},\underline{y})$ between two events $\underline{x}$ and $\underline{y}$ is defined by

(4.12) $$(\underline{x},\underline{y})^2 = (x_0-y_0)^2 - \sum_{i=1}^{3} (x_i-y_i)^2.$$

A nonsingular inhomogeneous transformation of M has the form

(4.13) $$\underline{x} \to T\underline{x} + \underline{y}, \qquad x \in M,$$

where T is a nonsingular operator on M, and $\underline{y}$ a fixed point in M. The *Poincaré group* P is defined to consist of those transformations (4.13) that respect the distance (4.12). Clearly, a nonsingular operator T belongs to P if and only if

(4.14) $$T^tFT = F,$$

where

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Notice that (4.14) is equivalent to the condition that T preserves the quadratic form

$$(\underline{x})^2 := x_0^2 - x_1^2 - x_2^2 - x_3^2, \qquad x \in M.$$

This is the so-called Minkowskian norm on M. From (4.14) it follows that if $T = (t_{ij})_{i,j=0}^{3}$ belongs to P, then (i) $\det(T) = \pm 1$ and (ii) $|t_{00}| \geq 1$. The subgroup of P consisting of all nonsingular operators T satisfying (4.14) is called the *Lorentz group* (or the homogeneous Poicaré group). General elements of P are denoted by $(\underline{y},T)$. Multiplication in P is given by

$$(\underline{y},T)(\underline{z},U) = (\underline{y}+T(\underline{z}),TU).$$

Notice that L is a lcsc. group since it is a closed subgroup of $G\ell(4,\mathbb{R})$. The mapping $(\underline{y},T) \to T(\underline{y})$ is clearly continuous in the product topology of $\mathbb{R}^4 \times L$, and therefore P is the semi-direct product of $N = \mathbb{R}^4$ (considered as a translation group) and $H = L$, relative to

$$\alpha_T(\underline{y}) = T(\underline{y}).$$

We are, however, at this moment merely interested in the connected component of the identity $(0,I)$ in P. Since $\mathbb{R}^4$ is already connected it suffices to look for the connected component of the identity in L. It can be shown that L consists of four connected components (cf. VARADARAJAN [21, thm.12.1]), which are given by

$$L_+^\uparrow = \{T \in L;\ \det(T) = \pm 1, \quad t_{00} \geq 1\};$$

$$L_-^\uparrow = \{T \in L;\ \det(T) = -1, \quad t_{00} \geq 1\};$$

$$L_+^\downarrow = \{T \in L;\ \det(T) = +1, \quad t_{00} \leq -1\};$$

$$L_-^\downarrow = \{T \in L;\ \det(T) = -1, \quad t_{00} \leq -1\}.$$

Of these sets; $L_+^\uparrow$ is the connected component of the identity, and therefore a closed invariant subgroup of L. The semi-direct product $\mathbb{R}^4 \circledS L_+^\uparrow$ is called the *continuous Poincaré group*, and denoted by $P_+^\uparrow$. The group $L_+^\uparrow$ is called the *orthochronous Lorentz group*. For computing the representations of $P_+^\uparrow$ it is rather convenient to compute those of its two-fold covering group.

It can be shown that the unimodular Lie group $S\ell(2,\mathbb{C})$ is the two-fold covering group of $L_+^\uparrow$. Since $\mathbb{R}^4$ and $S\ell(2,\mathbb{C})$ are both simply connected, their topological product is so, too. If Λ: $S\ell(2,\mathbb{C}) \to L_+^\uparrow$ denotes the two-to-one covering homomorphism, we can make the product $\mathbb{R}^4 \alpha\ S\ell(2,\mathbb{C})$ into a semi-direct product by setting

$$(\underline{x},A)(\underline{y},B) = (\underline{x}+\Lambda(A)\underline{y},AB),\ \underline{x},\underline{y} \in \mathbb{R}^4,\ \ A,B \in S\ell(2,\mathbb{C}).$$

Then the mapping $(\underline{x},A) \to (\underline{x},\Lambda(A))$ provides a two-to-one covering of $P_+^\uparrow$. For convenience, we recall how the mapping Λ: $S\ell(2,C) \to L_+^\uparrow$ is defined. Let $\sigma_0,\sigma_1,\sigma_2,\sigma_3$ denote the four Pauli matrices, defined by

$$\sigma_0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ \sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ \sigma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\ \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

With an element $\underline{x}$ of M we associate an hermitian 2×2-matrix $\hat{\underline{x}}$ by

$$(4.15)\qquad \hat{\underline{x}} := \sum_{i=1}^{3} x_i\sigma_i = \begin{pmatrix} x_0+x_3 & x_1-ix_2 \\ x_1+ix_2 & x_0-x_3 \end{pmatrix}.$$

It can be readily verified that the assignment $\underline{x} \to \hat{\underline{x}}$ is a linear isomorphism from $\mathbb{R}^4$ onto the space of all hermitian 2×2-matrices. Denote this space by H(2). If $A \in S\ell(2,\mathbb{C})$ and if A^* is the hermitian adjoint of A, then

$$X \to AXA^*, \qquad X \in H(2),$$

defines a linear one-to-one mapping from H(2) onto itself, which preserves determinants. Now we define an operator $\Lambda(A)$, $A \in S\ell(2,\mathbb{C})$, on M, by

$$\Lambda(A)(\underline{x}) = \underline{y}, \quad \text{where } \hat{\underline{y}} = A\hat{\underline{x}}A^*.$$

We contend that the operator $\Lambda(A)$ respects the distance (4.12). Indeed, straightforward calculation shows that the distance $(\underline{x},\underline{y})$ is equal to the square root of $\det(\underline{x}-\underline{y})^\wedge)$, and

$$\det(\Lambda(A)(\underline{x})^\wedge) = \det(\hat{\underline{x}}).$$

The character group of $\mathbb{R}^4$ is isomorphic with $\mathbb{R}^4$, and the characters are given by

$$\phi_{\underline{y}}(\underline{x}) = e^{i(\underline{x},\underline{y})}, \qquad \underline{y} \in \mathbb{R}^4,$$

where $(\underline{x},\underline{y}) = x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3$. The group $S\ell(2,\mathbb{C})$ acts on $\hat{\mathbb{R}}^4$ by

$$A[\phi_{\underline{y}}] = \phi_{\Lambda(A)\underline{y}}.$$

Indeed, since $\det(T) = 1$, for T in $L_+^\uparrow$, we have

$$(A[\phi_{\underline{y}}])(\underline{x}) = \phi_{\underline{y}}(\Lambda(A)^{-1}\underline{x}) = e^{i(\Lambda(A)^{-1}\underline{x},\underline{y})} = e^{i(\underline{x},\Lambda(A)\underline{y})} = \phi_{\Lambda(A)\underline{y}}(\underline{x}).$$

Orbits

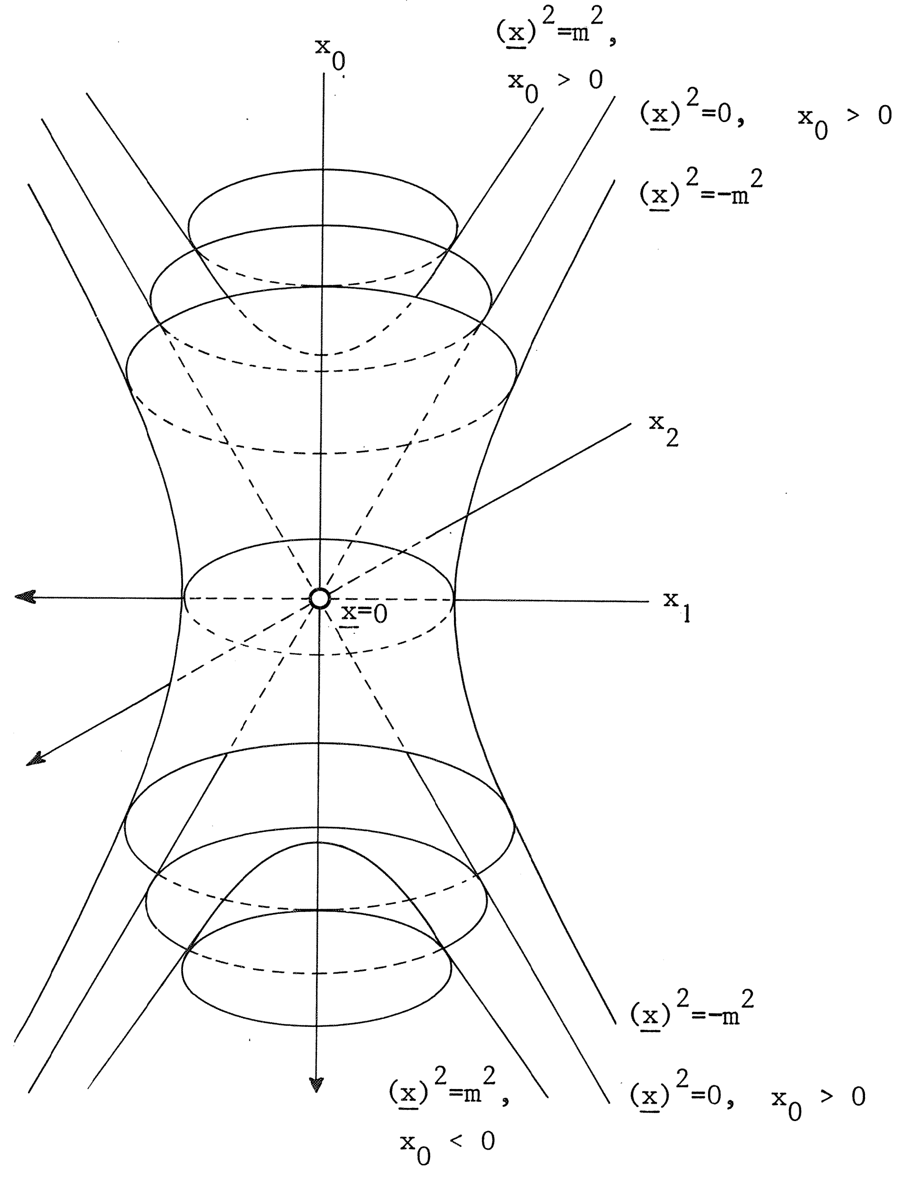
The orbits of $S\ell(2,\mathbb{C})$ in $\mathbb{R}^4$ are characterized in the first place by the relation $(\underline{x})^2$ = constant. That is, each set $\{\underline{x} \in \mathbb{R}^4;\ (\underline{x})^2 = m^2\}$, $m \in \mathbb{R}$, must be a union of orbits. There are three types of such sets:

$$\begin{aligned}
&(\underline{x})^2 = m^2,\ m > 0 &&: \text{two-sheeted hyperboloid;}\\
&(\underline{x})^2 = 0 &&: \text{cone;}\\
&(\underline{x})^2 = -m^2,\ m > 0 &&: \text{one-sheeted hyperboloid.}
\end{aligned}$$

Since $S\ell(2,\mathbb{C})$ is connected, its orbits in $\mathbb{R}^4$ have to be connected as well. Therefore, in the case $(\underline{x})^2 = m^2$, $m > 0$, each sheet of the hyperboloid is a union of orbits. Using concrete Lorentz transformations one can readily show that the sheets are actually orbits in their own right. As to the cone, it contains the trivial orbit ω_0, which splits it in two disconnected parts. Using straightforward arguments one proves transitivity of $S\ell(2,\mathbb{C})$ on the following sets:

$$\begin{aligned}
\tilde{\omega}_0 &:= \{0\};\\
\tilde{\omega}_m^+ &:= \{\underline{x} \in \mathbb{R}^4;\ (\underline{x})^2 = m^2,\ x_0 > 0\},\quad m > 0;\\
\tilde{\omega}_m^- &:= \{\underline{x} \in \mathbb{R}^4;\ (\underline{x})^2 = m^2,\ x_0 < 0\},\quad m > 0;\\
\tilde{\omega}_{im} &:= \{\underline{x} \in \mathbb{R}^4;\ (\underline{x})^2 = -m^2\},\ m > 0;\\
\tilde{\omega}_0^+ &:= \{x \in \mathbb{R}^4;\ (\underline{x})^2 = 0,\ x_0 > 0\};\\
\tilde{\omega}_0^- &:= \{x \in \mathbb{R}^4;\ (\underline{x})^2 = 0,\ x_0 < 0\}.
\end{aligned}$$

Consequently, these are the orbits of $S\ell(2,\mathbb{C})$ in $\mathbb{R}^4$. Accordingly, the orbits in $\hat{\mathbb{R}}^4$ are given by $\omega_m^+ := \{\phi_{\underline{x}};\ \underline{x} \in \tilde{\omega}_m^+\}$, etc. If we keep x_3 fixed, then it is possible to make an interesting drawing of the parametrization of the orbits, see figure 1.



x_0
$(\underline{x})^2=m^2$,
$x_0 > 0$
$(\underline{x})^2=0$, $x_0 > 0$
$(\underline{x})^2=-m^2$
x_2
x_1
$\underline{x}=0$
$(\underline{x})^2=-m^2$
$(\underline{x})^2=m^2$,
$x_0 < 0$
$(\underline{x})^2=0$, $x_0 > 0$

Figure 1

Stabilizers

Ad ω_0: $H_0 = S\ell(2,\mathbb{C})$.

Ad ω_m^+: Fix the point $\phi_{(m,0,0,0,)}$, and consider the stabilizer H_m^+ in $S\ell(2,\mathbb{C})$ of $(m,0,0,0)$. The corresponding matrix $\hat{\underline{x}}$ in $H(2)$ defined by (4.15) is

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}.$$

An element A of $S\ell(2,\mathbb{C})$ belongs to H_m^+ if and only if

$$\hat{\underline{x}} = A\hat{\underline{x}}A^*. \tag{4.16}$$

This equation is equivalent to $AA^* = I$, and therefore $H_m^+ = SU(2)$, the special unitarian group.

Ad ω_m^-: $H_m^- = H_m^+ = SU(2)$.

Ad ω_{im}: Consider the stabilizer H_{im} of $(0,0,m,0)$. It consists of all matrices A in $S\ell(2,\mathbb{C})$ which satisfy $\sigma_2 = A\sigma_2 A^*$. Since

$$(A^*)^{-1} = \sigma_2^{-1} A\sigma_2 = -\sigma_2^t A\sigma_2 = (A^t)^{-1},$$

this condition is equivalent to $A^* = A^t$. This is true if and only if A has real entries. Hence, $H_{im} = S\ell(2,\mathbb{R})$.

Ad ω_0^+: Fix $\underline{x} = (1,0,0,1)$, then

$$\hat{\underline{x}} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Identity (4.16) in this case is readily seen to be equivalent to

$$A = \begin{pmatrix} e^{i\theta} & z \\ 0 & e^{-i\theta} \end{pmatrix}, \quad \theta \in [0,2\pi),\ z \in \mathbb{C}. \tag{4.17}$$

Hence, the stabilizer H_0^+ is the group of all matrices of the form (4.17). We define

$$(z,\theta) := \begin{pmatrix} e^{2i\theta} & e^{-i\theta}z \\ 0 & e^{-2i\theta} \end{pmatrix}.$$

Then H_0^+ can be identified with the set $\{(z,\theta);\ z \in \mathbb{C}, \theta \in [0,2\pi)\}$, and its multiplication is given by

$$(z_1,\theta_1)(z_2,\theta_2) = (z_1+e^{2i\theta_1}z_2, \theta_1+\theta_2).$$

Therefore, H_0^+ is a semi-direct product of $\mathbb{C}$ and $\mathbb{T}$, the circle group, relative to

$$\alpha_\theta(z) = e^{2i\theta}z.$$

The Euclidean group $E(2)$ can also be considered as a semi-direct product of $\mathbb{C}$ and $\mathbb{T}$, with multiplication given by

$$[z_1,\theta_1][z_2,\theta_2] = [z_1+e^{i\theta_1}z_2, \theta_1+\theta_2].$$

Obviously, the mapping $(z,\theta) \longmapsto [z,2\theta]$ from H_0^+ onto $E(2)$ is a two-to-one homomorphism. Hence, we see that H_0^+ can be considered as a two-fold covering group of $E(2)$. This fact leads us to the notation $H_0^+ = \tilde{E}(2)$.

<u>Ad ω_0^-</u>: $H_0^- = H_0^+ = \tilde{E}(2)$.

One shows readily that $P_+^\uparrow$ is regular. Indeed, for each ordered pair (r_1,r_2) of rational numbers such that $0 < r_1 < r_2$, define three Borel subsets of $\hat{\mathbb{R}}^4$ by

$$B^\pm(r_1,r_2) := \bigcup_{r_1<m<r_2} \omega_m^\pm;\ B^i(r_1,r_2) = \bigcup_{r_1<m<r_2} \omega_{im}.$$

The collections of all such sets, complemented with the Borel sets ω_0, ω_0^+ and ω_0^-, is a countable family, which meets the requirements (4.11).

Consequently, the representation theory of $\tilde{P}_+^\uparrow$ (and hence that of $P_+^\uparrow$) is reduced to those of four smaller groups, $S\ell(2,\mathbb{C})$, $S\ell(2,\mathbb{R})$, $SU(2)$ and

$\tilde{E}(2)$. We proceed to classify the irreducible unitary representation associated with the orbits. The irreducible unitary representations of $S\ell(2,\mathbb{C})$ and $S\ell(2,\mathbb{R})$ are not discussed in these notes, and we will only state the results in these cases. For details, see for instance BARUT & RĄCZKA [2].

Ad ω_0: The set of irreducible unitary representation of $S\ell(2,\mathbb{C})$ consists of two series:

(i) the so-called *principal series*, parametrized by two numbers (r,j), $r \geq 0$, $j = 0,\frac{1}{2},1,\frac{3}{2},\ldots$

(ii) the so-called *supplementary series*, parametrized by a real number $r \in (-1,1)$, $r \neq 0$.

The extensions to $\tilde{P}_+^\uparrow$ are denoted by $\sigma^{(0,r,j)}$ and $\sigma^{(0,r)}$, respectively.

Ad $\omega_m^\pm$: As mentioned before, the set of irreducible unitary representations of $SU(2)$ consists of a series $D^{(s)}$, $s = 0,\frac{1}{2},\frac{3}{2},\ldots$, with $\dim(D^{(s)}) = 2s+1$. Hence, for each $m > 0$ we get two series of representations of $P_+^\uparrow$, associated with the orbits ω_m^+ and ω_m^-, respectively. We denote these series by $\sigma^{(m,\pm,s)}$.

Ad ω_{im}: The group $S\ell(2,\mathbb{R})$ has three series of irreducible unitary representations, which will be discussed in subsequent chapters. They are:

(i) the *principal series*, parametrized by two numbers (t,ε), $t \in \mathbb{R}$, $\varepsilon = 0$ or 1;

(ii) the *discrete series*, parametrized by integers, $n = 0,\pm1,\pm2,\ldots$

(iii) the *supplementary series*, parametrized by a real number $r \in (-1,1)$, $r \neq 0$.

The corresponding representations of $P_+^\uparrow$ are denoted by $\sigma^{(im,t,\varepsilon)}$, $\sigma^{(im,u)}$ and $\sigma^{(im,r)}$.

Ad $\omega_0^\pm$: We showed that $H_0^\pm = \tilde{E}(2)$ is the semi-direct product of $\mathbb{C}$ and $\mathbb{T}$, relative to $\alpha_\theta(z) = e^{2i\theta}z$. The character group $\hat{\mathbb{C}}$ is isomorphic with $\mathbb{C}$, and the characters are given by

$$\phi_w(z) = e^{i\mathrm{Re}(z\bar{w})}, \qquad w \in \mathbb{C}.$$

The circle group $\mathbb{T}$ acts on $\hat{\mathbb{C}}$ by

$$\theta[\phi_w] = \phi_{e^{2i\theta}w}, \qquad \theta \in [0,2\pi).$$

Hence, the orbits are circles in $\mathbb{C}$, which we denote by $\omega_r := \{\phi_z;\ |z| = r\}$, $r \geq 0$. The irreducible unitary representations of $\tilde{E}(2)$ associated with ω_0 are those of $\mathbb{T}$, extended to $\tilde{E}(2)$. We denote them by L^j, $j = 0,\pm 1,\pm 2,\ldots$.

The stabilizer in $\mathbb{T}$ of ϕ_r, $r > 0$, is $\{0,\pi\}$. This cyclic group has two irreducible representations on $\mathbb{C}$:

$$\psi^0 : 0,\pi \to 1, \qquad \psi^1 : 0 \to 1, \qquad \pi \to -1.$$

We denote the corresponding representations of $\tilde{E}(2)$ by $L^{r,\varepsilon}$ with $\varepsilon = 0$ or 1.

For $P_+^\uparrow$ we find the following two series of irreducible unitary representations, associated with the orbits $\omega_0^\pm$:

$$\sigma^{(\omega_0^\pm, L^j)} =: \sigma^{(0,\pm,j)}, \qquad j = 0,\pm 1,\pm 2,\ldots$$

$$\sigma^{(\omega_0^\pm, L^{r,\varepsilon})} := \sigma^{(0,\pm,r,\varepsilon)}, \quad r > 0,\ \varepsilon = 0 \text{ or } 1.$$

THEOREM 4.13. *The set of irreducible unitary representations of the continuous Poincaré group consists (up to equivalence) of the following eight series:*

(i) $\sigma^{(0,r,j)}$; $r \geq 0$, $j = 0,\frac{1}{2},1,\frac{3}{2},\ldots$;

(ii) $\sigma^{(0,r)}$, $-1 < r < 1$, $r \neq 0$;

(iii) $\sigma^{(m,\pm,s)}$, $s = 0,\frac{1}{2},1,\frac{3}{2},\ldots$;

(iv) $\sigma^{(im,t,\varepsilon)}$, $t \in \mathbb{R}$, $\varepsilon = 0$ or 1;

(v) $\sigma^{(im,n)}$, $n = 0,\pm 1,\pm 2,\ldots$;

(vi) $\sigma^{(im,r)}$, $-1 < r < 1$, $r \neq 0$;

(vii) $\sigma^{(0,\pm,j)}$ $j = 0,\pm 1,\pm 2,\ldots$;

(viii) $\sigma^{(0,\pm,r,\varepsilon)}$, $r > 0$, $\varepsilon = 0$ or 1.

Forced by lack of space-time we can not discuss the physical interpretations and the explicit realizations of those series of representations. A few suggestions to the reader for finding information on these aspects, are: BARUT & RĄCZKA [2], SIMMS [20] and VARADARAJAN [21].

Appendix to §4.3.

- *Let* A *be a lcsc. abelian group, and suppose that we are given a finite complex-valued Borel measure on the dual group* A, *such that*

$$\text{(A.4)} \qquad \int_{\hat{A}} \phi(a)d\mu(\phi) = 0 \qquad \forall a \in A.$$

Then $\mu = 0$.

PROOF. Consider the space $L^1(A)$, consisting of all Haar-summable complex-valued functions on A. The *Fourier transform* $\hat{f}$ of a function $f \in L^1(A)$ is a function on $\hat{A}$, defined by

$$\hat{f}(\phi) := \int_A f(a)\phi(a)da, \qquad \phi \in \hat{A}.$$

It can be shown that $f \to \hat{f}$ maps $L^1(A)$ onto a dense subalgebra of $C_0(\hat{A})$, the space of complex-valued continuous functions on $\hat{A}$ which vanish at infinity. (cf. REITER [15, §5.4.2], the topology on $C_0(\hat{A})$ is the usual one; generated by the sup-norm). By an application of the Fubini theorem, we find

$$\int_{\hat{A}} \hat{f}(\phi)d\mu(\phi) = \int_{\hat{A}}\int_A f(a)\phi(a)da d\mu(\phi) =$$

$$= \int_A f(a)\Big(\int_{\hat{A}} \phi(a)d\mu(\phi)\Big)da, \qquad f \in L^1(A).$$

Hence, using (A.4) and a density argument, we find

$$\int_{\hat{A}} g(\phi)d\mu(\phi) = 0 \qquad \forall g \in C_0(\hat{A}).$$

This implies that μ is equal to zero (cf. RUDIN [17, thm.6.19]). □

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